Exact solution and perturbation theory in a general quantum system

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By splitting a Hamiltonian into two parts, using the solvability of eigenvalue problem of one part of the Hamiltonian, proving a useful identity and deducing an expansion formula of power of operator binomials, we obtain an explicit and general form of time evolution operator in the representation of solvable part of the Hamiltonian. Further we find out an exact solution of Schrödinger equation in a general time-independent quantum system, and write down its concrete form when the solvable part of this Hamiltonian is taken as the kinetic energy term. Comparing our exact solution with the usual perturbation theory makes some features and significance of our solution clear. Moreover, through deriving out the improved forms of the zeroth, first, second and third order perturbed solutions including the partial contributions from the higher order even all order approximations, we obtain the improved transition probability. In special, we propose the revised Fermi's golden rule. Then we apply our scheme to obtain the improved forms of perturbed energy and perturbed state. In addition, we study an easy understanding example to illustrate our scheme and show its advantage. All of this implies the physical reasons and evidences why our exact solution and perturbative scheme are formally explicit, actually calculable, operationally efficient, conclusively more accurate. Therefore our exact solution and perturbative scheme can be thought of theoretical developments of quantum dynamics. Further applications of our results in quantum theory can be expected.

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I. INTRODUCTION

One of the most important tasks of physics is to obtain the time evolution law and its form of a physical system, that is, so-called dynamical equation and its solution. In a quantum system, the dynamical equation is the Schrödinger equation [1, 2] for a pure state or the von Neumann equation for a mixed state [3]. Suppose that, at time $t = 0$, an arbitrary initial state is $|\Psi(0)\rangle$, then at a given time t, the state of quantum system will become

$$|\Psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht}|\Psi(0)\rangle. \tag{1}$$

where H is the Hamiltonian of a quantum system and it is assumed independent of time, while $\mathrm{e}^{-\mathrm{i}Ht}$ ($\hbar = 1$) is called as the time evolution operator. Above equation describes an arbitrary initial state how to evolve to a final state, that is, it is a formal solution of the Schrödinger equation for a pure state. Unfortunately, only finite several quantum systems are exactly solvable at present. This implies that in a general quantum system, the time evolution of expanding coefficients of final state in some given representation can not be clearly and explicitly expressed by the c-number function. From our point of view, this shortcoming leads to that we can not fully determine and exactly calculate those physical quantities dependent on the state (that is expanding coefficients in a given representation) at any time. In other words, above formal solution (1) of final state in the operator form is not really useful for the many practical purposes. For example, in the cases with environment-system interaction, how does the final state decohere, and in the cases with an entangled initial state, how does the entanglement of final state vary, and so on?

In order to overcome above shortcoming appearing in a general quantum system, quantum perturbation theory is well developed and it is successfully applied the calculations of perturbed energy, perturbed state and transition probability etc. In the usual perturbation theory, the key idea to research the time evolution of system is to split the Hamiltonian of system into two parts, that is

$$H = H_0 + H_1, \tag{2}$$

where the eigenvalue problem of H_0 is solvable, and H_1 is the rest part of the Hamiltonian except for H_0. In other words, this splitting is chosen in such a manner that the solutions of H_0 are known as

$$H_0|\Phi^\gamma\rangle = E_\gamma|\Phi^\gamma\rangle, \tag{3}$$

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where $|\Phi^\gamma\rangle$ is the eigenvector of H_0 and E_γ is the corresponding eigenvalue. However, when H_1 is not so small compared with H_0 that we have to consider the higher order approximations or when the partial contributions from higher order even all order approximations become relatively important to the studied problems, the usual perturbation theory might be difficult to calculate to an enough precision in an effective way, even not feasible practically since the lower approximation might break the physical symmetries and/or constraints. In our point of view, the physical reasons resulting in above difficulties and obstacles are that the usual perturbation theory does not give a general term form of expanding coefficient evolution with time for any order approximation and does not explicitly express it as a c-number function. Thus, it is necessary to find the perturbed solution from the low to the high order approximation step by step up to some order approximation for a needed precision. These problems motivate us to focus on finding an explicit expression of general term for any order approximation, that is an exact solution that is not in the operator form, and then reasonably and physically deduce out a scheme of perturbation theory. In fact, as soon as this purpose is arrived at, we will obtain an exact solution with all order contributions coming from H_1. This consequentially implies, in principle, that this scheme can more effectively and more accurately solve the physical problems than using the existing method. Then, we concretely realize and demonstrate this physical idea by providing a scheme of perturbation theory. In this process, just like the accustomed way to study a more complicated quantum system, if H_1 is indeed small enough compared with H_0, its influence may be as a perturbation and one can take up to some order approximation conveniently at a needed precision. However, the distinct difference between ours and usual one is that our perturbed solution will be able to include the partial contributions from the higher order even all order approximations based on the reasonable physical considerations so that the physical problems are more accurately and effectively calculated, and kept their symmetries and/or constraints as possible since we have used the feature of our exact solution with the general term form of time evolution of arbitrary initial state and added systematically the contributions from the high order approximations.

Actually, one of interesting and important tasks in the study of quantum dynamics is to find clearly the the expanding coefficients (amplitudes) before the basis vectors in a given representation how to evolve. The typical examples are the calculations decoherence and disentanglement of quantum systems. Although we split the Hamiltonian into two parts in terms of the known method, the problem has not yet been finally answered because that H_0 does not commute with H_1 in general, and the matrix elements of time evolution operator $\mathrm{e}^{-\mathrm{i}Ht}$ do not have the closed factor forms about time t in the given representations including H_0 representation spanned by $\{|\Phi^\gamma\rangle, \gamma=1,2,\cdots\}$ in terms of the known perturbation theory. To solve this kind of problems is also one of our main motivations.

Just well-known, quantum dynamics and its perturbation theory are correct and have been sufficiently studied. Many famous physicists created their nice formulism and obtained some marvelous results. An attempt to improve its part content or increase some new content as well as some new methods must be very difficult in their realizations. This results in our this possibly lengthy manuscript. However, our endeavor is only at the beginning. It must be pointed out we still use the extensively accepted principles and laws of quantum mechanics, and we try to use elementary mathematics as possible. All of mathematics knowledge used here does not exceed the university level.

Undoubtedly, the formalization of physical theory including quantum dynamics often has its highly mathematical focus, but this can not cover its real motivations, potential applications and related conclusions in physics. We study the reexpression in form should not be only a mathematical game and skill, we expect to find its physical content and applications. This is leaded actually by our more explicit and general form of the Schrödinger equation and our perturbative scheme. A series of conclusions done here make us assure that we have partially arrived at our purpose.

In this paper, we start from finding an explicitly closed form about time t of matrix elements of time evolution operator $\mathrm{e}^{-\mathrm{i}Ht}$ in the H_0 representation spanned by $\{|\Phi^\gamma\rangle, \gamma=1,2,\cdots\}$. Our method is first to derive out an expansion formula of power of operator binomials and then apply it to the Taylor's expansion of time evolution operator $\mathrm{e}^{-\mathrm{i}Ht}$. Moreover, by proving and using our identity, we derive out the explicit and general form of representation matrix of $\mathrm{e}^{-\mathrm{i}Ht}$ in the representation of H_0. Consequently, we obtain an exact solution of the Schrödinger equation in a general time-independent quantum system, in particular, its concrete form when the solvable part of Hamiltonian is taken as the kinetic energy term. Furthermore, by comparing our solution with the usual perturbation theory, we reveal their relation and show what is more as well as what is different in our exact solution. The conclusions clearly indicate that our exact solution is consistent with the usual perturbation theory at any order approximation, but also in our exact solution we explicitly calculate out the expanding coefficients of unperturbed state in Lippmann-Schwinger equation [4] for the time-independent (stationary) perturbation theory and/or fully solves the recurrence equation of the expansion coefficients of final state in H_0 representation from a view of time-dependent (dynamical) perturbation theory. After introducing the perturbative method, in order to provide a new scheme of perturbation theory based on our exact solution, we first propose two useful skills. Then, we expressly demonstrate our solution is not only formally explicit, but also actually more accurate and effective via generally deriving out the improved forms of the zeroth, first, second and third order approximation of perturbed solution including partial contributions from the higher order even all order approximations, finding the improved transition probability, specially, the revised Fermi's golden rule, and providing a operational scheme to calculate the perturbed energy and perturbed state. Furthermore, by studying

a concrete example of two state system, we illustrate clearly that our solution is more efficient and more accurate than the usual perturbative method. In short, our exact solution and perturbative scheme are formally explicit, actually calculable, operationally efficient, conclusively more accurate (to the needed precision).

It is worth emphasizing that we obtain an explicit and general expression of time evolution operator that is a summation over all order contributions from the rest part of Hamiltonian except for the solvable part, or all order approximations of the perturbative part of the Hamiltonian. This makes us be able to build our perturbative scheme via so-called "dynamical rearrangement and summation" technology, which is seen in Sec. VII. In a sentence, our perturbative scheme comes from the our exact solution, and then show our exact solution indeed contain useful and interesting physical content besides its explicit mathematics form.

Based on the above all of reasons, our exact solution and perturbative scheme can be thought of theoretical developments of quantum dynamics. In our point of view, it is helpful for understanding the dynamical behavior and related subjects of quantum systems in theory. Specially, we think that the features and advantages of our solution can not be fully revealed only by the perturbative method, because it is an exact solution. We would like to study the possible applications to the formulation of quantum dynamics in the near future.

This paper consists of 11 sections and one appendix. Besides Sec. I is an introduction, Sec. XI is the conclusion and discussion, the other 9 sections can be divided into three parts. The first part made of sections from two to four is to deduce and prove our exact solution in the general time-independent quantum system; the second part made of section five is to compare our exact solution including all of order approximations with the usual perturbation theory and demonstrate the features and advantages of our exact solution; the third part made of sections from six to ten is to propose our perturbative scheme, obtain its application and illustrate its results. More concretely, they are organized as the following: in Sec. II we first propose the expansion formula of power of operator binomials; in Sec. III we derive out an explicit and general expression of time evolution operator by proving and using our identity; in Sec. IV we obtain the exact solution of the Schrödinger equation and present a concrete example when the solvable part of Hamiltonian is taken as the kinetic energy term; in Sec. V we compare our solution with the usual perturbation theory and prove their consistency, their relations and explain what is more as well as what is different in our solution; in Sec. VI we introduce two skills to relate our exact solution and the perturbative scheme in order to include the contributions from the high order approximations; in Sec. VII we deduce out the improved perturbed solution of dynamics including partial contributions from the higher even all order approximations; in Sec. VIII we obtain the improved transition probability, specially, the revised Fermi's golden rule. In Sec. IX we provide a scheme to calculate the perturbed energy and the perturbed state; in Sec. X we study an example of two state system in order to concretely illustrate our solution to be more effective and more accurate than the usual method; in Sec. XI we summarize our conclusions and give some discussions. Finally, we write an appendix where the proof of our identity is presented and some expressions are calculated in order to derive out our improved forms of perturbed solutions.

II. EXPANSION FORMULA OF POWER OF OPERATOR BINOMIALS

In order to obtain the explicitly exact solution of the Schrödinger equation in a general time-independent quantum system, let us start with the derivation of expansion formula of power of operator binomials. Without loss of generality, we are always able to write the power of operator binomials as two parts

$$(A+B)^n = A^n + f^n(A,B). \tag{4}$$

where A and B are two operators and do not commute with each other in general. While the second part $f^n(A,B)$ is a polynomials including at least first power of B and at most nth power of B in its every term. Thus, a general term with lth power of B has the form $\left(\prod_{i=1}^{l} A^{k_i} B\right) A^{n-l-\sum_{i=1}^{l} k_i}$. From the symmetry of power of binomials we conclude that every k_i take the values from 0 to $(n-l)$, but it must keep $n-l-\sum_{i=1}^{l} k_i \geq 0$. So we have

$$f^n(A,B) = \sum_{l=1}^{n} \sum_{\substack{k_1,\cdots,k_l=0 \\ \sum_{i=1}^{l} k_i + l \leq n}}^{n-l} \left(\prod_{i=1}^{l} A^{k_i} B\right) A^{n-l-\sum_{i=1}^{l} k_i} \tag{5}$$

$$= \sum_{l=1}^{n} \sum_{k_1,\cdots,k_l=0}^{n-l} \left(\prod_{i=1}^{l} A^{k_i} B\right) A^{n-l-\sum_{i=1}^{l} k_i} \theta\left(n-l-\sum_{i=1}^{l} k_i\right). \tag{6}$$

where $\theta(x)$ is a step function, that is, $\theta(x)=1$ if $x\geq 0$, and $\theta(x)=0$ if $x<0$. Obviously based on above definition, we easily verify

$$f^1(A,B)=B,\quad f^2(A,B)=AB+B(A+B). \tag{7}$$

They imply that the expression (5) is correct for $n=1,2$.

Now we use the mathematical induction to proof the expression (5) of $f^n(A,B)$, that is, let us assume that it is valid for a given n, and then prove that it is valid too for $n+1$. Denoting $\mathcal{F}^{n+1}(A,B)$ with the form of expression (6) where n is replaced by $n+1$, and we extract the part of $l=1$ in its finite summation for l

$$\begin{aligned}\mathcal{F}^{n+1}(A,B) &= \sum_{k_1=0}^{n} A^{k_1}BA^{n-k_1}+\sum_{l=2}^{n+1}\sum_{k_1,\cdots,k_l=0}^{(n+1)-l}\left(\prod_{i=1}^{l}A^{k_i}B\right)A^{(n+1)-l-\sum_{i=1}^{l}k_i}\\ &\times\theta\left((n+1)-l-\sum_{i=1}^{l}k_i\right).\end{aligned} \tag{8}$$

We extract the terms $k_1=0$ in the first and second summations, and again replace k_1 by k_1-1 in the summation (the summations for k_i from 1 to $(n+1)-l$ change as one from 0 to $n-l$), the result is

$$\begin{aligned}\mathcal{F}^{n+1}(A,B) &= BA^n+\sum_{k_1=1}^{n}A^{k_1}BA^{n-k_1}+B\sum_{l=2}^{n+1}\sum_{k_2,\cdots,k_l=0}^{(n+1)-l}\left(\prod_{i=2}^{l}A^{k_i}B\right)\\ &\times A^{(n+1)-l-\sum_{i=2}^{l}k_i}\theta\left((n+1)-l-\sum_{i=2}^{l}k_i\right)\\ &+\sum_{l=2}^{n+1}\sum_{k_1=1}^{(n+1)-l}\sum_{k_2,\cdots,k_l=0}^{(n+1)-l}\left(\prod_{i=1}^{l}A^{k_i}B\right)\\ &\times A^{(n+1)-l-\sum_{i=1}^{l}k_i}\theta\left((n+1)-l-\sum_{i=1}^{l}k_i\right),\end{aligned} \tag{9}$$

furthermore

$$\begin{aligned}\mathcal{F}^{n+1}(A,B) &= BA^n+A\sum_{k_1=0}^{n-1}A^{k_1}BA^{n-1-k_1}+B\sum_{l=2}^{n+1}\sum_{k_2,\cdots,k_l=0}^{(n+1)-l}\left(\prod_{i=2}^{l}A^{k_i}B\right)\\ &\times A^{(n+1)-l-\sum_{i=2}^{l}k_i}\theta\left((n+1)-l-\sum_{i=2}^{l}k_i\right)\\ &+A\sum_{l=2}^{n+1}\sum_{k_1=0}^{n-l}\sum_{k_2,\cdots,k_l=0}^{(n+1)-l}\left(\prod_{i=1}^{l}A^{k_i}B\right)A^{n-l-\sum_{i=1}^{l}k_i}\theta\left(n-l-\sum_{i=1}^{l}k_i\right).\end{aligned} \tag{10}$$

Considering the third term in above expression, we change the dummy index $\{k_2,k_3,\cdots,k_l\}$ into $\{k_1,k_2,\cdots,k_{l-1}\}$, rewrite $(n+1)-l$ as $n-(l-1)$, and finally replace l by $l-1$ in the summation (the summation for l from 2 to $n+1$ changes as one from 1 to n), we obtain

$$\begin{aligned}&B\sum_{l=2}^{n+1}\sum_{k_2,\cdots,k_l=0}^{(n+1)-l}\left(\prod_{i=2}^{l}A^{k_i}B\right)A^{(n+1)-l-\sum_{i=2}^{l}k_i}\theta\left((n+1)-l-\sum_{i=2}^{l}k_i\right)\\ &=B\sum_{l=2}^{n+1}\sum_{k_1,\cdots,k_{l-1}=0}^{n-(l-1)}\left(\prod_{i=1}^{l-1}A^{k_i}B\right)A^{n-(l-1)-\sum_{i=1}^{l-1}k_i}\theta\left(n-(l-1)-\sum_{i=1}^{l-1}k_i\right)\\ &=B\sum_{l=1}^{n}\sum_{k_1,\cdots,k_l=0}^{n-l}\left(\prod_{i=1}^{l}A^{k_i}B\right)A^{n-l-\sum_{i=1}^{l}k_i}\theta\left(n-l-\sum_{i=1}^{l}k_i\right)\\ &=Bf^n(A,B),\end{aligned} \tag{11}$$

where we have used the expression (6) of $f^n(A,B)$. Because of the step function $\theta\left(n-l-\sum_{i=1}^{l}k_i\right)$ in the fourth term of the expression (10), the upper bound of summation for l is abated to n, and the upper bound of summation for $k_2,k_3,\cdots,k_l$ is abated to $n-l$. Then, merging the fourth term and the second term in eq.(10) gives

$$\begin{aligned}
& A\sum_{k_1=0}^{n-1}A^{k_1}BA^{n-1-k_1}+A\sum_{l=2}^{n+1}\sum_{k_1=0}^{n-l}\sum_{k_2,\cdots,k_l=0}^{(n+1)-l}\left(\prod_{i=1}^{l}A^{k_i}B\right)\\
&\quad\times A^{n-l-\sum_{i=1}^{l}k_i}\theta\left(n-l-\sum_{i=1}^{l}k_i\right)\\
&= A\sum_{k_1=0}^{n-1}A^{k_1}BA^{n-1-k_1}+A\sum_{l=2}^{n}\sum_{k_1,k_2,\cdots,k_l=0}^{n-l}\left(\prod_{i=1}^{l}A^{k_i}B\right)\\
&\quad\times A^{n-l-\sum_{i=1}^{l}k_i}\theta\left(n-l-\sum_{i=1}^{l}k_i\right)\\
&= A\sum_{l=1}^{n}\sum_{k_1,k_2,\cdots,k_l=0}^{n-l}\left(\prod_{i=1}^{l}A^{k_i}B\right)A^{n-l-\sum_{i=1}^{l}k_i}\theta\left(n-l-\sum_{i=1}^{l}k_i\right)\\
&= Af^n(A,B),
\end{aligned}\tag{12}$$

where we have used again the expression (6) of $f^n(A,B)$.

Substituting (11) and (12) into (10), immediately leads to the following result

$$\mathcal{F}^{n+1}=BA^n+(A+B)f^n(A,B).\tag{13}$$

Note that

$$\begin{aligned}
(A+B)^{n+1} &= (A+B)(A+B)^n\\
&= A^{n+1}+BA^n+(A+B)f^n(A,B),
\end{aligned}\tag{14}$$

we have the relation

$$f^{n+1}(A,B)=BA^n+(A+B)f^n(A,B).\tag{15}$$

Therefore, in terms of eqs.(13) and (15) we have finished our proof that $f^n(A,B)$ has the expression (5) or (6) for any n.

III. EXPRESSION OF THE TIME EVOLUTION OPERATOR

Now we investigate the expression of the time evolution operator by means of above expansion formula of power of operator binomials, that is, we write

$$e^{-\mathrm{i}Ht}=\sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^n}{n!}(H_0+H_1)^n=\mathrm{e}^{-\mathrm{i}H_0t}+\sum_{n=0}^{\infty}\frac{(-\mathrm{i}t)^n}{n!}f^n(H_0,H_1).\tag{16}$$

In above equation, inserting the complete relation $\sum_\gamma|\Phi^\gamma\rangle\langle\Phi^\gamma|=1$ before every $H_0^{k_i}$, and using the eigen equation of H_0 (3), it is easy to see that

$$\begin{aligned}
f^n(H_0,H_1) &= \sum_{l=1}^{n}\sum_{\gamma_1,\cdots,\gamma_{l+1}}\sum_{\substack{k_1,\cdots,k_l=0\\ \sum_{i=1}^{l}k_i+l\le n}}^{n-l}\left[\prod_{i=1}^{l}E_{\gamma_i}^{k_i}\right]\\
&\quad\times E_{\gamma_{l+1}}^{n-\sum_{i=1}^{l}k_i-l}\left[\prod_{i=1}^{l}H_1^{\gamma_i\gamma_{i+1}}\right]|\Phi^{\gamma_1}\rangle\langle\Phi^{\gamma_{l+1}}|\\
&= \sum_{l=1}^{n}\sum_{\gamma_1,\cdots,\gamma_{l+1}}C_l^n(E[\gamma,l])\left[\prod_{i=1}^{l}H_1^{\gamma_i\gamma_{i+1}}\right]|\Phi^{\gamma_1}\rangle\langle\Phi^{\gamma_{l+1}}|,
\end{aligned}\tag{17}$$

where $H_1^{\gamma_i\gamma_{i+1}} = \langle\Phi^{\gamma_i}|H_1|\Phi^{\gamma_{i+1}}\rangle$, $E[\gamma,l]$ is a vector with $l+1$ components denoted by

$$E[\gamma,l] = \{E_{\gamma_1}, E_{\gamma_2}, \cdots, E_{\gamma_l}, E_{\gamma_{l+1}}\} \tag{18}$$

and we introduce the definition of $C_l^n(E[\gamma,l])$ $(l \geq 1)$ as the following

$$C_l^n(E[\gamma,l]) = \sum_{\substack{k_1,\cdots,k_l=0\\ \sum_{i=1}^{l} k_i+l\leq n}}^{n-l} \left[\prod_{i=1}^{l} E_{\gamma_i}^{k_i}\right] E_{\gamma_{l+1}}^{n-\sum_{i=1}^{l} k_i - l}. \tag{19}$$

In order to derive out an explicit and useful expression of $C_l^n(E[\gamma,l])$, we first change the dummy index $k_l \to n - l - \sum_{i=1}^{l} k_i$ in the eq.(6). Note that at the same time in spite of $\theta(n-l-\sum_{i=1}^{l} k_i)$ changes as $\theta(k_l)$, but a hiding factor $\theta(k_l)$ becomes $\theta(n-l-\sum_{i=1}^{l} k_i)$. Thus, we can rewrite

$$\begin{aligned} f^n(A,B) \; &= \sum_{l=1}^{n} \sum_{k_1,\cdots,k_l=0}^{n-l} \left(\prod_{i=1}^{l-1} A^{k_i} B\right) A^{n-l-\sum_{i=1}^{l} k_i} B \\ &\quad \times A^{k_l} \theta\left(n - l - \sum_{i=1}^{l} k_i\right). \end{aligned} \tag{20}$$

In fact, this new expression of $f^n(A,B)$ is a result of the symmetry of power of binomials for its every factor. Similarly, in terms of the the above method to obtain the definition of $C_l^n(E[\gamma,l])$ (19), we have its new definition (where k_l is replaced by k)

$$\begin{aligned} C_l^n(E[\gamma,l]) \; &= \sum_{k=0}^{n-l} \left\{ \sum_{k_1,\cdots,k_{l-1}=0}^{n-l} \left[\prod_{i=1}^{l-1} E_{\gamma_i}^{k_i}\right] E_{\gamma_l}^{(n-k)-\sum_{i=1}^{l-1} k_i - l} \right. \\ &\quad \left. \times \theta\left((n-k) - l - \sum_{i=1}^{l-1} k_i\right)\right\} E_{\gamma_{l+1}}^{k}. \end{aligned} \tag{21}$$

Because that $\theta\left((n-k) - l - \sum_{i=1}^{l-1} k_i\right)$ abates the upper bound of summation for $k_i (i = 1, 2, \cdots, l-1)$ from $(n-l)$ to $(n-k-1)-(l-1)$, we obtain the recurrence equation

$$\begin{aligned} C_l^n(E[\gamma,l]) \; &= \sum_{k=0}^{n-l} \left\{ \sum_{k_1,\cdots,k_{l-1}=0}^{(n-k-1)-(l-1)} \left[\prod_{i=1}^{l-1} E_{\gamma_i}^{k_i}\right] E_{\gamma_l}^{(n-k-1)-\sum_{i=1}^{l-1} k_i - (l-1)} \right. \\ &\quad \left. \times \theta\left((n-k-1) - (l-1) - \sum_{i=1}^{l-1} k_i\right)\right\} E_{\gamma_{l+1}}^{k} \\ &= \sum_{k=0}^{n-l} C_{l-1}^{n-k-1}(E[\gamma,l-1]) E_{\gamma_{l+1}}^{k}. \end{aligned} \tag{22}$$

In particular, when $l = 1$, from the definition of $C_l^n(E(\gamma,1))$ it follows that

$$C_1^n(E(\gamma,1)) = \sum_{k_1=0}^{n-1} E_{\gamma_1}^{k_1} E_{\gamma_2}^{n-1-k_1} = E_{\gamma_2}^{n-1} \sum_{k_1=0}^{n-1} \left(\frac{E_{\gamma_1}}{E_{\gamma_2}}\right)^{k_1}. \tag{23}$$

By means of the summation formula of a geometric series, we find that

$$C_1^n(E(\gamma,1)) = \frac{E_{\gamma_1}^n}{E_{\gamma_1} - E_{\gamma_2}} - \frac{E_{\gamma_2}^n}{E_{\gamma_1} - E_{\gamma_2}}. \tag{24}$$

Based on the recurrence equation (22), we have

$$
\begin{aligned}
C_2^n(E(\gamma,2)) &= \sum_{k=0}^{n-2}\left(\frac{E_{\gamma_1}^{n-k-1}}{E_{\gamma_1}-E_{\gamma_2}}-\frac{E_{\gamma_2}^{n-k-1}}{E_{\gamma_1}-E_{\gamma_2}}\right)E_{\gamma_3}^k \\
&= \sum_{i=1}^{2}\frac{(-1)^{i-1}E_{\gamma_i}^{n-1}}{E_{\gamma_1}-E_{\gamma_2}}\sum_{k=0}^{n-2}\left(\frac{E_{\gamma_3}}{E_{\gamma_i}}\right)^k \\
&= \sum_{i=1}^{2}\frac{(-1)^{i-1}(E_{\gamma_i}^n-E_{\gamma_i}E_{\gamma_3}^{n-1})}{(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_i}-E_{\gamma_3})} \\
&= \frac{E_{\gamma_1}^n}{(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma_3})}-\frac{E_{\gamma_2}^n}{(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_2}-E_{\gamma_3})} \\
&\quad +\frac{E_{\gamma_3}^n}{(E_{\gamma_1}-E_{\gamma_3})(E_{\gamma_2}-E_{\gamma_3})}.
\end{aligned} \tag{25}
$$

In the above calculations, the last step is important. In fact, in order to obtain the concrete expression of C_l^n (l and n are both positive integers), we need our identity

$$
\sum_{i=1}^{l+1}(-1)^{i-1}\frac{E_{\gamma_i}^K}{d_i(E[\gamma,l])}=\begin{cases}0 & (\text{If } 0\le K<l)\\ 1 & (\text{If } K=l)\end{cases}. \tag{26}
$$

It is proved in Appendix A in detail. The denominators $d_i(E[\gamma,l])$ in above identity are defined by

$$
\begin{aligned}
d_1(E[\gamma,l]) &= \prod_{i=1}^{l}\left(E_{\gamma_1}-E_{\gamma_{i+1}}\right), && (27)\\
d_i(E[\gamma,l]) &= \prod_{j=1}^{i-1}\left(E_{\gamma_j}-E_{\gamma_i}\right)\prod_{k=i+1}^{l+1}\left(E_{\gamma_i}-E_{\gamma_k}\right), && (28)\\
d_{l+1}(E[\gamma,l]) &= \prod_{i=1}^{l}\left(E_{\gamma_i}-E_{\gamma_{l+1}}\right), && (29)
\end{aligned}
$$

where $2\le i\le l$. Then, using the recurrence equation (22) and our identity (26), we obtain

$$
C_l^n(E[\gamma,l])=\sum_{i=1}^{l+1}(-1)^{i-1}\frac{E_{\gamma_i}^n}{d_i(E[\gamma,l])}. \tag{30}
$$

Here, the mathematical induction shows its power again. If the expression (30) is correct for a given n, for example $n=1,2$, then for $n+1$ from the recurrence equation (22) it follows that

$$
\begin{aligned}
C_l^{n+1}(E[\gamma,l]) &= \sum_{k=0}^{n+1-l}\left(\sum_{i=1}^{l}(-1)^{i-1}\frac{E_{\gamma_i}^{n-k}}{d_i(E[\gamma,l-1])}\right)E_{\gamma_{l+1}}^k \\
&= \sum_{i=1}^{l}\frac{(-1)^{i-1}E_{\gamma_i}^n}{d_i(E[\gamma,l-1])}\sum_{k=0}^{n+1-l}\left(\frac{E_{\gamma_{l+1}}}{E_{\gamma_i}}\right)^k \\
&= \sum_{i=1}^{l}\frac{(-1)^{i-1}(E_{\gamma_i}^{n+1}-E_{\gamma_i}^{l-1}E_{\gamma_{l+1}}^{n+2-l})}{d_i(E[\gamma,l-1])(E_{\gamma_i}-E_{\gamma_{l+1}})}.
\end{aligned} \tag{31}
$$

Because that $d_i(E[\gamma,l-1])(E_{\gamma_i}-E_{\gamma_{l+1}})=d_i(E[\gamma,l])$ ($i\le l+1$) and our identity (26), it is easy to see

$$
\sum_{i=1}^{l}\frac{(-1)^{i-1}E_{\gamma_i}^{l-1}}{d_i(E[\gamma,l])}=-(-1)^l\frac{E_{\gamma_{l+1}}^{l-1}}{d_{l+1}(E[\gamma,l])}. \tag{32}
$$

Substitute it into eq.(31) yields

$$C_l^{n+1}(E[\gamma,l])=\sum_{i=1}^{l+1}(-1)^{i-1}\frac{E_{\gamma_i}^{n+1}}{d_i(E[\gamma,l])}. \tag{33}$$

That is, we have proved that the expression (30) of $C_l^n(E[\gamma,l])$ is valid for any n.

It must be emphasized that for the diagonal elements or in the degeneration cases, we need to understand above expressions in the sense of limitations. For instance, for $C_1^n(E[\gamma,1])=\left(E_{\gamma_1}^n-E_{\gamma_2}^n\right)/\left(E_{\gamma_1}-E_{\gamma_2}\right)$, we have that the expression $\lim_{E_{\gamma_2}\to E_{\gamma_1}} C_1^n(E[\gamma,1])=\left(\delta_{\gamma_1\gamma_2}+\Theta(|\gamma_2-\gamma_1|)\delta_{E_{\gamma_1}E_{\gamma_2}}\right)nE_{\gamma_1}^{n-1}$, where the step function $\Theta(x)=1$, if $x>0$, and $\Theta(x)=0$, if $x\leq 0$. Because of our identity (26), the summation to l in eq.(17) can be extended to ∞. Thus, the expression of the time evolution operator is changed to a summation according to the order (or power) of the H_1 as the following

$$\left\langle\Phi^\gamma\middle|\mathrm{e}^{-\mathrm{i}Ht}\middle|\Phi^{\gamma'}\right\rangle=\mathrm{e}^{-\mathrm{i}E_\gamma t}\delta_{\gamma\gamma'}+\sum_{l=1}^{\infty}\sum_{\gamma_1,\cdots,\gamma_{l+1}}\left[\sum_{i=1}^{l+1}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,l])}\right]\prod_{j=1}^{l}H_1^{\gamma_j\gamma_{j+1}}\delta_{\gamma_1\gamma}\delta_{\gamma_{l+1}\gamma'}. \tag{34}$$

It is clear that it has the closed time evolution factors.

IV. SOLUTION OF THE SCHRÖDINGER EQUATION

Substituting the our expression of the time evolution operator (34) into eq.(1), we obtain immediately our explicit form of time evolution of an arbitrary initial state in a general quantum system

$$|\Psi(t)\rangle = \sum_{\gamma,\gamma'}\left[\mathrm{e}^{-\mathrm{i}E_\gamma t}\delta_{\gamma\gamma'}+\sum_{l=1}^{\infty}A_l^{\gamma\gamma'}(t)\right]\left[\langle\Phi^{\gamma'}|\Psi(0)\rangle\right]|\Phi^\gamma\rangle, \tag{35}$$

$$A_l^{\gamma\gamma'}(t) = \sum_{\gamma_1,\cdots,\gamma_{l+1}}\left[\sum_{i=1}^{l+1}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,l])}\right]\left[\prod_{j=1}^{l}H_1^{\gamma_j\gamma_{j+1}}\right]\delta_{\gamma_1\gamma}\delta_{\gamma_{l+1}\gamma'}. \tag{36}$$

Of course, since the linearity of the evolution operator and the completeness of eigenvector set of H_0, we can simply set the initial state as an eigenvector of H_0. Up to now, everything is exact and no any approximation enters. Therefore, it is an exact solution in spite of its form is an infinity series. In other words, it exactly includes all order approximations of H_1. Only when H_1 is taken as a perturbation, it can be cut-off based on the needed precision. From a view of formalized theory, it is explicit and general, but is not compact. Moreover, if the convergence is guaranteed, it is strict since its general term is known. Usually, to a practical purpose, if only including the finite (often low) order approximation of H_1, above expression is cut-off to the finite terms and becomes a perturbed solution.

Actually, since our solution includes all order contributions from the rest part of Hamiltonian except for the solvable part H_0, or all order approximations of the perturbative part H_1 of Hamiltonian, it is not very important whether H_1 is (relatively) large or small when compared with H_0. In principle, for a general quantum system with the normal form Hamiltonian, we always can write down

$$H=\frac{\hat{\boldsymbol{k}}^2}{2m}+V. \tag{37}$$

If taking the solvable kinetic energy term as $H_0=\dfrac{\hat{\boldsymbol{k}}^2}{2m}$ and the potential energy part as $H_1=V$, our method is applicable to such quantum system. It is clear that

$$\frac{\hat{\boldsymbol{k}}^2}{2m}|\boldsymbol{k}\rangle=\frac{\boldsymbol{k}^2}{2m}|\boldsymbol{k}\rangle=E_{\boldsymbol{k}}|\boldsymbol{k}\rangle,\quad E_{\boldsymbol{k}}=\frac{\boldsymbol{k}^2}{2m}. \tag{38}$$

And denoting

$$\langle\boldsymbol{x}|\boldsymbol{k}\rangle = \frac{1}{L^{3/2}}\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{39}$$

$$\psi_{\boldsymbol{k}}(0) = \langle\boldsymbol{k}|\Psi(0)\rangle, \tag{40}$$

$$\Psi(\boldsymbol{x},t) = \langle\boldsymbol{x}|\Psi(t)\rangle, \tag{41}$$

we obtain the final state as

$$\begin{aligned}\Psi(\boldsymbol{x},t) &= \sum_{\boldsymbol{k}}\psi_{\boldsymbol{k}}(0)\frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-E_{\boldsymbol{k}}t)}}{L^{3/2}} \\ &\quad+\sum_{l=1}^{\infty}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{\boldsymbol{k}_1,\cdots,\boldsymbol{k}_{l+1}}\psi_{\boldsymbol{k}'}(0)\left[\prod_{j=1}^{l}V^{\boldsymbol{k}_j\boldsymbol{k}_{j+1}}\right]\delta_{\boldsymbol{k}_1\boldsymbol{k}}\delta_{\boldsymbol{k}_{l+1}\boldsymbol{k}'} \\ &\quad\times\left[\sum_{i=1}^{l+1}\frac{(-1)^{i-1}}{d_i(E[k,l])}\,\frac{1}{L^{3/2}}\,\mathrm{e}^{\mathrm{i}\left(\boldsymbol{k}\cdot\boldsymbol{x}-E_{\boldsymbol{k}_i}t\right)}\right] \qquad (42)\\ &= \sum_{\boldsymbol{k}}\psi_{\boldsymbol{k}}(0)\frac{\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-E_{\boldsymbol{k}}t)}}{L^{3/2}} \\ &\quad+\sum_{l=1}^{\infty}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{\boldsymbol{k}_1,\cdots,\boldsymbol{k}_{l+1}}\psi_{\boldsymbol{k}'}(0)\left[\prod_{j=1}^{l}\mathcal{V}(\boldsymbol{k}_j-\boldsymbol{k}_{j+1})\right]\delta_{\boldsymbol{k}_1\boldsymbol{k}}\delta_{\boldsymbol{k}_{l+1}\boldsymbol{k}'} \\ &\quad\times\left[\sum_{i=1}^{l+1}\frac{(-1)^{i-1}}{d_i(E[k,l])}\,\frac{1}{L^{3/2}}\,\mathrm{e}^{\mathrm{i}\left(\boldsymbol{k}\cdot\boldsymbol{x}-E_{\boldsymbol{k}_i}t\right)}\right]. \qquad (43)\end{aligned}$$

In the second equal mark, we have used the fact that $V=V(\boldsymbol{x})$ usually, thus

$$\begin{aligned}V^{\boldsymbol{k}_j\boldsymbol{k}_{j+1}} &= \langle\boldsymbol{k_j}|V(\boldsymbol{x})|\boldsymbol{k}_{j+1}\rangle \\ &= \int_{-\infty}^{\infty}\mathrm{d}^3x_j\mathrm{d}^3x_{j+1}\langle\boldsymbol{k_j}|\boldsymbol{x}_j\rangle\langle\boldsymbol{x}_j|V(\boldsymbol{x})|\boldsymbol{x}_{j+1}\rangle\langle\boldsymbol{x}_{j+1}|\boldsymbol{k_{j+1}}\rangle \\ &= \frac{1}{L^3}\int_{-\infty}^{\infty}\mathrm{d}^3x_j\mathrm{d}^3x_{j+1}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}_j\cdot\boldsymbol{x}_j}V(\boldsymbol{x}_{j+1})\delta^3(\boldsymbol{x}_j-\boldsymbol{x}_{j+1})\mathrm{e}^{\mathrm{i}\boldsymbol{k}_{j+1}\cdot\boldsymbol{x}_{+1}j} \\ &= \frac{1}{L^3}\int_{-\infty}^{\infty}\mathrm{d}^3xV(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}(\boldsymbol{k_j}-\boldsymbol{k_{j+1}})\cdot\boldsymbol{x}} \\ &= \mathcal{V}(\boldsymbol{k}_j-\boldsymbol{k}_{j+1}). \qquad (44)\end{aligned}$$

It is clear that $\mathcal{V}(\boldsymbol{k})$ is the Fourier transformation of $V(\boldsymbol{x})$. The eq.(42) or (43) is a concrete form of our solution (35) of Schrödinger equation when the solvable part of Hamiltonian is taken as the kinetic energy term. Therefore, in principle, if the Fourier transformation of $V(\boldsymbol{x})$ can be found, the evolution of the arbitrarily initial state with time can be obtained. It must be emphasized that it is often that the solvable part of Hamiltonian will be not only $T=\dfrac{\hat{\boldsymbol{k}}^2}{2m}$ for the practical purposes. Moreover, since the general term is obtained, we can selectively include the partial contributions from some high order even all order approximations.

Furthermore, we can derive out the propagator

$$\begin{aligned}\langle\boldsymbol{x}|\mathrm{e}^{-\mathrm{i}Ht}|\boldsymbol{x}'\rangle &= \sum_{\boldsymbol{k},\boldsymbol{k}'}\langle\boldsymbol{x}|\boldsymbol{k}\rangle\langle\boldsymbol{k}|\mathrm{e}^{-\mathrm{i}Ht}|\boldsymbol{k}'\rangle\langle\boldsymbol{k}'|\boldsymbol{x}'\rangle \\ &= \sum_{\boldsymbol{k}}\frac{1}{L^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')-\mathrm{i}E_{\boldsymbol{k}}t}+\sum_{l=1}^{\infty}\sum_{\boldsymbol{k},\boldsymbol{k}'}\sum_{\boldsymbol{k}_1,\cdots,\boldsymbol{k}_{l+1}}\left[\prod_{j=1}^{l}V^{\boldsymbol{k}_j\boldsymbol{k}_{j+1}}\right]\delta_{\boldsymbol{k}_1\boldsymbol{k}}\delta_{\boldsymbol{k}_{l+1}\boldsymbol{k}'} \\ &\quad\times\left[\sum_{i=1}^{l+1}\frac{(-1)^{i-1}}{d_i(E[k,l])}\,\frac{1}{L^{3/2}}\,\mathrm{e}^{-\mathrm{i}E_{\boldsymbol{k}_i}t}\right]\frac{1}{L^3}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}-\mathrm{i}\boldsymbol{k}'\cdot\boldsymbol{x}'}. \qquad (45)\end{aligned}$$

It is different from the known propagator in the expressive form, but they should be equivalent to the physical results. One of its distinguished features is this matrix element to be fully c-number function made of the matrix elements $V^{\gamma_i\gamma_j}$ and unperturbed energy levels E_{γ_i}.

V. COMPARING WITH THE USUAL PERTURBATION THEORY

In this section, we will compare the usual perturbation theory with our solution, reveal their consistency and relation, and point out what is more in our solution and what is different among them. Furthermore, we expect to

reveal the features, significance and possible applications of our solution in theory. We will respectively investigate and discuss the cases comparing with the time-independent, time-dependent perturbation theories as well as the non-perturbed solution.

A. Comparing with the time-independent perturbation theory

The usual time-independent (stationary) perturbation theory is mainly to study the stationary wave equation in order to obtain the perturbative energies and perturbative states. However, our solution focus on the development of quantum states, which is a solution of the Schrödinger dynamical equation. Although their main purposes are different at their start point, but they are consistent. Moreover, our exact solution is also able to apply to the calculation of perturbed energy and perturbed state, which will be seen clearly in Sec. IX.

In order to verify that the time-independent perturbation theory is consistent with our solution, we suppose that in the initial state, the system is in the eigen state of the total Hamiltonian, that is

$$H|\Psi_{E_T}(0)\rangle = E_T|\Psi_{E_T}(0)\rangle. \tag{46}$$

It is clear that

$$|\Psi_{E_T}(t)\rangle = \mathrm{e}^{-\mathrm{i}E_T t}|\Psi_{E_T}(0)\rangle = \sum_{\gamma,\gamma'}\left\{\mathrm{e}^{-\mathrm{i}E_\gamma t}\delta_{\gamma\gamma'} + \sum_{l=1}^{\infty} A_l^{\gamma\gamma'}(t)\right\}\left[\langle\Phi^{\gamma'}|\Psi_{E_T}(0)\rangle\right]|\Phi^\gamma\rangle. \tag{47}$$

To calculate the Kth time derivative of this equation and then set $t=0$, we obtain

$$\begin{aligned} {E_T}^K|\Psi_{E_T}(0)\rangle &= \sum_{\gamma,\gamma'}\left\{E_\gamma^K\delta_{\gamma\gamma'} + (\mathrm{i})^K\sum_{l=1}^{\infty}\left.\frac{\mathrm{d}^K A_l^{\gamma\gamma'}(t)}{\mathrm{d}t^K}\right|_{t=0}\right\}a_{\gamma'}|\Phi^\gamma\rangle & (48)\\ &= \sum_{\gamma,\gamma'}\left\{E_\gamma^K\delta_{\gamma\gamma'} + \sum_{l=1}^{\infty}B_l^{\gamma\gamma'}(K)\right\}a_{\gamma'}|\Phi^\gamma\rangle, & (49)\end{aligned}$$

where we have used the fact that

$$|\Psi_{E_T}(0)\rangle = \sum_\gamma a_\gamma|\Phi^\gamma\rangle, \quad a_\gamma = \langle\Phi^\gamma|\Psi_{E_T}(0)\rangle, \tag{50}$$

$$\left.\frac{\mathrm{d}^K A_l^{\gamma\gamma'}(t)}{\mathrm{d}t^K}\right|_{t=0} = (-\mathrm{i})^K B_l^{\gamma\gamma'}(K). \tag{51}$$

It is easy to obtain that

$$\begin{aligned} \left.\frac{\mathrm{d}^K A_l^{\gamma\gamma'}(t)}{\mathrm{d}t^K}\right|_{t=0} &= (-\mathrm{i})^K\sum_{\gamma_1,\cdots,\gamma_{l+1}}\left[\sum_{i=1}^{l+1}(-1)^{i-1}\frac{E_{\gamma_i}^K}{d_i(E[\gamma,l])}\right]\left[\prod_{j=1}^{l}H_1^{\gamma_j\gamma_{j+1}}\right]\delta_{\gamma_1\gamma}\delta_{\gamma_{l+1}\gamma'}, & (52)\\ B_l^{\gamma\gamma'}(K) &= \sum_{\gamma_1,\cdots,\gamma_{l+1}} C_l^K(E[\gamma,l])\theta(K-l)\left[\prod_{j=1}^{l}H_1^{\gamma_j\gamma_{j+1}}\right]\delta_{\gamma_1\gamma}\delta_{\gamma_{l+1}\gamma'}. & (53)\end{aligned}$$

where we have used our identity (26). This means that if $l>K$

$$\left.\frac{\mathrm{d}^K A_{l>K}^{\gamma\gamma'}(t)}{\mathrm{d}t^K}\right|_{t=0} = (-\mathrm{i})^K B_l^{\gamma\gamma'}(K>l) = 0. \tag{54}$$

In special

$$\left.\frac{\mathrm{d}A_l^{\gamma\gamma'}(t)}{\mathrm{d}t}\right|_{t=0} = -\mathrm{i}B_1^{\gamma\gamma'}(1) = -\mathrm{i}H_1^{\gamma\gamma'}\delta_{l1}. \tag{55}$$

$$\left.\frac{\mathrm{d}^2 A_l^{\gamma\gamma'}(t)}{\mathrm{d}t^2}\right|_{t=0} = -B_l^{\gamma\gamma'}(2) = -\left(E_\gamma + E_{\gamma'}\right) H_1^{\gamma\gamma'}\delta_{l1} - \sum_{\gamma_1} H_1^{\gamma\gamma_1} H_1^{\gamma_1\gamma'}\delta_{l2}. \tag{56}$$

In other words, the summation to l in the right side of eq.(48) is cut-off to K. Therefore, by left multiplying $\langle\Phi^\gamma|$ to eq.(48) we obtain

$$E_T{}^K a_\gamma = E_\gamma^K a_\gamma + \sum_{\gamma'}\sum_{l=1}^{K} B_l^{\gamma\gamma'}(K) a_{\gamma'}. \tag{57}$$

When $K=1$, above equation backs to the start point of time-independent perturbation theory

$$E_T a_\gamma = E_\gamma a_\gamma + \sum_{\gamma'} H_1^{\gamma\gamma'} a'_\gamma. \tag{58}$$

It implies that it is consistent with our solution. In order to certify that our solution is compatible with the time-independent perturbation theory, we have to consider the cases when $K \geq 2$. Note that $E_T^K a_\gamma = E_T\left(E_T^{K-1} a_\gamma\right)$, we substitute eq.(57) twice, then move the second term to the right side, we have

$$\begin{aligned} E_\gamma^{K-1} E_T a_\gamma &= E_\gamma^K a_\gamma + \sum_{\gamma'}\sum_{l=1}^{K} B_l^{\gamma\gamma'}(K) a_{\gamma'} - \sum_{\gamma'}\sum_{l=1}^{K-1} B_l^{\gamma\gamma'}(K-1)\sum_{\gamma''}\left(E_{\gamma'}\delta_{\gamma'\gamma''} + H_1^{\gamma'\gamma''}\right) a_{\gamma''} \\ &= E_\gamma^K a_\gamma + \sum_{\gamma'}\sum_{l=1}^{K-1}\left[B_l^{\gamma\gamma'}(K) - E_{\gamma'} B_l^{\gamma\gamma'}(K-1)\right] a_{\gamma'} \\ &\quad + \sum_{\gamma'}\left(\prod_{j=1}^{K} H_1^{\gamma_j\gamma_{j+1}}\right)\delta_{\gamma\gamma_1}\delta_{\gamma_{l+1}\gamma'} a_{\gamma'} - \sum_{\gamma'}\sum_{l=1}^{K-1} B_l^{\gamma\gamma'}(K-1)\sum_{\gamma''} H_1^{\gamma'\gamma''} a_{\gamma''}. \end{aligned} \tag{59}$$

Since when $K \geq 2$

$$C_1^K(E[\gamma,1]) - E_{\gamma_2} C_1^{K-1}(E[\gamma,1]) = E_{\gamma_1}^{K-1}, \tag{60}$$

and again $l \geq 2$

$$C_l^K(E[\gamma,l]) - E_{\gamma_{l+1}} C_l^{K-1}(E[\gamma,l]) = C_{l-1}^{K-1}(E[\gamma,l-1]), \tag{61}$$

which has been proved in appendix A. Obviously, when $K=2$, eq.(59) becomes

$$E_\gamma E_T a_\gamma = E_\gamma^2 a_\gamma + E_\gamma \sum_{\gamma'} H_1^{\gamma\gamma'} a_{\gamma'}. \tag{62}$$

It can back to eq.(58). Similarly, for $K \geq 3$, so do they. In fact, from eqs.(60,61), eq.(59) becomes

$$E_\gamma^{K-1} E_T a_\gamma \;=\; E_\gamma^K a_\gamma + E_\gamma^{K-1}\sum_{\gamma'} H_1^{\gamma\gamma'} a_{\gamma'}, \tag{63}$$

where we have used the fact

$$\begin{aligned} &\sum_{\gamma'}\sum_{l=2}^{K-1}\left[B_l^{\gamma\gamma'}(K) - E_{\gamma'} B_l^{\gamma\gamma'}(K-1)\right] a_{\gamma'} + \sum_{\gamma'}\left(\prod_{j=1}^{K} H_1^{\gamma_j\gamma_{j+1}}\right)\delta_{\gamma\gamma_1}\delta_{\gamma_{l+1}\gamma'} a_{\gamma'} \\ &- \sum_{\gamma'}\sum_{l=1}^{K-1} B_l^{\gamma\gamma'}(K-1)\sum_{\gamma''} H_1^{\gamma'\gamma''} a_{\gamma''} = 0. \end{aligned} \tag{64}$$

Its proof is not difficult because we can derive out

$$
\begin{aligned}
&\sum_{\gamma'}\sum_{l=2}^{K-1}\left[B_l^{\gamma\gamma'}(K)-E_{\gamma'}B_l^{\gamma\gamma'}(K-1)\right]a_{\gamma'}\\
&=\sum_{\gamma'}\sum_{l=2}^{K-1}\sum_{\gamma_1,\cdots,\gamma_{l+1}}\left[C_l^K(E[\gamma,l])-E_{\gamma'}C_l^{K-1}(E[\gamma,l])\right]\left(\prod_{j=1}^{l}H_1^{\gamma_j\gamma_{j+1}}\right)\delta_{\gamma\gamma_1}\delta_{\gamma_{l+1}\gamma'}a_{\gamma'}\\
&=\sum_{\gamma'}\sum_{l=2}^{K-1}\sum_{\gamma_1,\cdots,\gamma_{l+1}}C_{l-1}^{K-1}(E[\gamma,l-1])\left(\prod_{j=1}^{l}H_1^{\gamma_j\gamma_{j+1}}\right)\delta_{\gamma\gamma_1}\delta_{\gamma_{l+1}\gamma'}a_{\gamma'}\\
&=\sum_{\gamma'}\sum_{l=1}^{K-2}\sum_{\gamma_1,\cdots,\gamma_{l+1}}C_l^{K-1}(E[\gamma,l])\left(\prod_{j=1}^{l}H_1^{\gamma_j\gamma_{j+1}}\right)H_1^{\gamma_{l+1}\gamma'}\delta_{\gamma\gamma_1}a_{\gamma'}\\
&=\sum_{\gamma',\gamma'}\sum_{l=1}^{K-2}B_l^{\gamma\gamma'}(K-1)H_1^{\gamma'\gamma''}a_{\gamma''}.
\end{aligned}
\tag{65}
$$

The first equality has used the definitions of $B_l^{\gamma\gamma'}(K)$, the second equality has used eq.(61), the third equality sets $l-1\to l$ and sums over γ_{l+2}, the forth equality sets $\gamma'\to\gamma''$, inserts $\sum_{\gamma'}\delta_{\gamma_{l+1}\gamma'}$ and uses the definition of $B_l^{\gamma\gamma'}(K)$ again. Substituting it into the left side of eq.(64) and using $B_{K-1}^{\gamma\gamma'}(K-1)$ expression, we can finish the proof of eq.(64). Actually, above proof further verify the correctness of our solution from the usual perturbation theory.

In our point of view, only if the expression of the high order approximation has been obtained, we can consider its contribution in the time-independent perturbation theory since its method is to find exactly a given order approximation. However, this task needs to solve a simultaneous equation system, it will be heavy for the enough high order approximation. In Sec. VII, we will give a method to find the improved forms of perturbed energy and perturbed state including partially high order even all order approximations.

Now, let us see what is more in our solution than the nonperturbative method (sometime be called time-independent theory). Actually, since H is not explicitly time-dependent, its eigenvectors can be given formally by so-called Lippmann-Schwinger equations as follows [4]:

$$
\begin{aligned}
|\Psi_{\rm S}^{\gamma}(\pm)\rangle &= |\Phi^\gamma\rangle+\frac{1}{E^\gamma-H_0\pm{\rm i}\eta}H_1|\Psi_{\rm S}^{\gamma}(\pm)\rangle &(66)\\
&= |\Phi^\gamma\rangle+G_0^\gamma(\pm)H_1|\Psi_{\rm S}^{\gamma}(\pm)\rangle &(67)\\
&= |\Phi^\gamma\rangle+\frac{1}{E_\gamma-H\pm{\rm i}\eta}H_1|\Phi^\gamma\rangle &(68)\\
&= |\Phi^\gamma\rangle+G^\gamma(\pm)H_1|\Phi^\gamma\rangle, &(69)
\end{aligned}
$$

where the complete Green's function $G^\gamma(\pm)=1/(E_\gamma-H\pm{\rm i}\eta)$ and the unperturbed Green's function $G_0^\gamma(\pm)=1/(E^\gamma-H_0\pm{\rm i}\eta)$ satisfy the Dyson's equation

$$
\begin{aligned}
G^\gamma(\pm) &= G_0^\gamma(\pm)+G_0^\gamma(\pm)H_1G^\gamma(\pm) &(70)\\
&= \sum_{l=0}^{\infty}\left(G_0^\gamma(\pm)H_1\right)^lG_0^\gamma(\pm). &(71)
\end{aligned}
$$

Here, the subscript"S" means the stationary solution:

$$
H|\Psi_{\rm S}^{\gamma}(\pm)\rangle=E_\gamma|\Psi_{\rm S}^{\gamma}(\pm)\rangle. \tag{72}
$$

More strictly, we should use the "in" and "out" states to express it [5]. In historical literature, this solution is known as so-called non-perturbative one. It has played an important role in the formal scatter theory.

Back to our attempt, for a given initial state $|\Psi(0)\rangle$, we have

$$
|\Psi(0)\rangle = \sum_{\gamma'}\langle\Psi_{\rm S}^{\gamma'}(\pm)|\Psi(0)\rangle\Big|\Psi_{\rm S}^{\gamma'}(\pm)\Big\rangle. \tag{73}
$$

Acting the time evolution operator on it, we immediately obtain

$$\begin{aligned}|\Psi(t)\rangle &= \sum_{\gamma'} \langle \Psi_{\rm S}^{\gamma'}(\pm)|\Psi(0)\rangle \mathrm{e}^{-\mathrm{i}E_{\gamma'}t}\left|\Psi_{\rm S}^{\gamma'}(\pm)\right\rangle \\ &= \sum_{\gamma'} \left\langle \Phi^{\gamma'}\right| \sum_{l=0}^{\infty}\left(H_1 G_0^{\gamma'}(\mp)\right)^l |\Psi(0)\rangle \mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \sum_{l'=0}^{\infty}\left(G^{\gamma'}(\pm)H_1\right)^{l'}\left|\Phi^{\gamma'}\right\rangle \\ &= \sum_{\gamma,\gamma'}\Bigg[\mathrm{e}^{-\mathrm{i}E_{\gamma}t}\delta_{\gamma\gamma'}\langle \Phi^{\gamma'}|\Psi(0)\rangle \\ &\quad + \sum_{\substack{l,l'=0\\ l+l'\neq 0}}^{\infty}\left\langle \Phi^{\gamma'}\right|\left(H_1 G_0^{\gamma'}(\mp)\right)^l|\Psi(0)\rangle\langle\Phi^{\gamma}|\left(G_0^{\gamma'}(\pm)H_1\right)^{l'}\left|\Phi^{\gamma'}\right\rangle \mathrm{e}^{-\mathrm{i}E_{\gamma'}t}\Bigg]|\Phi^{\gamma}\rangle,\end{aligned} \tag{74}$$

comparing this result with our solution (35), we can say that our solution has finished the calculations of explicit form of the expanding coefficients (matrix elements) in the second term of the above equation using our method and rearrange the resulting terms according with the power of elements of representation matrix $H_1^{\gamma_j\gamma_{j+1}}$. It implies that our solution can have more and more explicit physical content and significance. Therefore, we think that our solution is a new development of the stationary perturbation theory. In spite of its physical results which consistent with the Feynman's diagram expansion of Dyson's equation, its form is new, explicit, and convenient to calculate the time evolution of states with time. Specially, it will be seen that our perturbative scheme based on our exact solution has higher efficiency and higher precision in the calculation from Secs. VIII, IX and X.

B. Comparing with the time-dependent perturbation theory

The aim of our solution is similar to the time-dependent perturbation theory. But their methods are different. The usual time-dependent perturbation theory [2] considers a quantum state initially in the eigenvector of the Hamiltonian H_0, and a subsequent evolution of system caused by the application of an explicitly time-dependent potential $V = \lambda v(t)$, where λ is enough small to indicate V as a perturbation, that is,

$$\mathrm{i}\frac{\partial}{\partial t}|\Psi(t)\rangle = (H_0 + \lambda v(t))|\Psi(t)\rangle \tag{75}$$

with the initial state

$$|\Psi(0)\rangle = |\Phi^{\alpha}\rangle. \tag{76}$$

It is often used to study the cases that the system returns to an eigenvector $|\Phi_f\rangle$ of the Hamiltonian H_0 when the action of the perturbing potential becomes negligible.

In order to compare it with our solution, let us first recall the time-dependent perturbation method. Note that the state $|\Psi(t)\rangle$ at time t can be expanded by the complete orthogonal system of the eigenvectors $|\Phi^{\gamma}\rangle$ of the Hamiltonian H_0, that is

$$|\Psi(t)\rangle = \sum_{\gamma} c_{\gamma}(t)|\Phi^{\gamma}\rangle \tag{77}$$

with

$$c_{\gamma}(t) = \langle \Phi^{\gamma}|\Psi(t)\rangle, \tag{78}$$

whereas the evolution equation (75) will become

$$\mathrm{i}\frac{\partial}{\partial t}c_{\gamma}(t) = E_{\gamma}c_{\gamma}(t) + \lambda\sum_{\gamma'} v^{\gamma\gamma'}(t)c_{\gamma'}(t), \tag{79}$$

where $v^{\gamma\gamma'}(t) = \langle \Phi^{\gamma}|v(t)\left|\Phi^{\gamma'}\right\rangle$. Now setting

$$c_{\gamma}(t) = b_{\gamma}(t)\mathrm{e}^{-\mathrm{i}E_{\gamma}t}, \tag{80}$$

and inserting this into (79) give

$$\mathrm{i}\frac{\partial}{\partial t}b_\gamma(t)=\lambda\sum_{\gamma'}\mathrm{e}^{\mathrm{i}(E_\gamma-E_{\gamma'})t}v^{\gamma\gamma'}(t)b_{\gamma'}(t). \tag{81}$$

Then, making a series expansion of $b_\gamma(t)$ according to the power of λ:

$$b_\gamma(t)=\sum_{l=0}\lambda^l b_\gamma^{(l)}(t), \tag{82}$$

and setting equal the coefficients of λ^l on the both sides of the equation (81), ones find:

$$\mathrm{i}\frac{\partial}{\partial t}b_\gamma^{(0)}(t) \;=\; 0, \tag{83}$$

$$\mathrm{i}\frac{\partial}{\partial t}b_\gamma^{(l)}(t) \;=\; \sum_{\gamma'}\mathrm{e}^{\mathrm{i}(E_\gamma-E_{\gamma'})t}v^{\gamma\gamma'}(t)b_{\gamma'}^{(l-1)}(t), \quad (l\neq 0). \tag{84}$$

From the initial condition (76), it follows that $c_\gamma(0)=b_\gamma(0)=\delta_{\gamma\alpha}=b_\gamma^{(0)}(0)$ and $b_\gamma^{(l)}(0)=0$ if $l\geq 1$. By integration over the variable t this will yield

$$b_\gamma^{(0)}(t) \;=\; \delta_{\gamma\alpha}, \tag{85}$$

$$b_\gamma^{(1)}(t) \;=\; -\mathrm{i}\int_0^t \mathrm{d}t_1\mathrm{e}^{\mathrm{i}(E_\gamma-E_\alpha)t_1}v^{\gamma\alpha}(t_1), \tag{86}$$

$$\begin{aligned}b_\gamma^{(2)}(t) \;&=\; -\mathrm{i}\int_0^t \mathrm{d}t_2\sum_{\gamma_2}\mathrm{e}^{\mathrm{i}(E_\gamma-E_{\gamma_2})t_2}v^{\gamma\gamma_2}(t_2)b_{\gamma_2}^{(1)}(t_2)\\ &=\; -\sum_{\gamma_2}\int_0^t\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_1\mathrm{e}^{\mathrm{i}(E_\gamma-E_{\gamma_2})t_2}\mathrm{e}^{\mathrm{i}(E_{\gamma_2}-E_\alpha)t_1}v^{\gamma\gamma_2}(t_2)v^{\gamma_2\alpha}(t_1).\end{aligned} \tag{87}$$

If we take v independent of time, the above method in the time-dependent perturbation theory is still valid. Thus,

$$b_\gamma^{(1)}(t)=-\frac{1}{E_\gamma-E_\alpha}\left(\mathrm{e}^{-\mathrm{i}E_\alpha t}-\mathrm{e}^{-\mathrm{i}E_\gamma t}\right)\mathrm{e}^{\mathrm{i}E_\gamma t}v^{\gamma\alpha}, \tag{88}$$

$$\begin{aligned}b_\gamma^{(2)}(t) \;&=\; \sum_{\gamma_2}\frac{1}{(E_\gamma-E_\alpha)(E_{\gamma_2}-E_\alpha)}\left(\mathrm{e}^{-\mathrm{i}E_\alpha t}-\mathrm{e}^{-\mathrm{i}E_\gamma t}\right)\mathrm{e}^{\mathrm{i}E_\gamma t}v^{\gamma\gamma_2}v^{\gamma_2\alpha}\\ &\quad -\sum_{\gamma_2}\frac{1}{(E_{\gamma_2}-E_\alpha)(E_\gamma-E_{\gamma_2})}\left(\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}-\mathrm{e}^{-\mathrm{i}E_\gamma t}\right)\mathrm{e}^{\mathrm{i}E_\gamma t}v^{\gamma\gamma_2}v^{\gamma_2\alpha}\\ &=\; \sum_{\gamma_2}\left[\frac{1}{(E_\gamma-E_{\gamma_2})(E_\gamma-E_\alpha)}-\frac{1}{(E_\gamma-E_{\gamma_2})(E_{\gamma_2}-E_\alpha)}\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}\mathrm{e}^{\mathrm{i}E_\gamma t}\right.\\ &\quad\left.+\frac{1}{(E_\gamma-E_\alpha)(E_{\gamma_2}-E_\alpha)}\mathrm{e}^{-\mathrm{i}E_\alpha t}\mathrm{e}^{\mathrm{i}E_\gamma t}\right]v^{\gamma\gamma_2}v^{\gamma_2\alpha}.\end{aligned} \tag{89}$$

This means that

$$c_\gamma^{(1)}(t)=\sum_{\gamma_1\gamma_2}\sum_{i=1}^{2}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}}}{d_i(E[\gamma,1])}v^{\gamma_1\gamma_2}\delta_{\gamma\gamma_1}\delta_{\gamma_2\alpha}, \tag{90}$$

$$c_\gamma^{(2)}(t)=\sum_{\gamma_1,\gamma_2,\gamma_3}\sum_{i=1}^{3}(-1)^{i-1}\frac{1}{d_i(E[\gamma,2])}\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}\delta_{\gamma\gamma_1}\delta_{\gamma_3\alpha}v^{\gamma_1\gamma_2}v^{\gamma_2\gamma_3}. \tag{91}$$

Now, we use the mathematical induction to prove

$$c_\gamma^{(l)}(t)=\sum_{\gamma_1,\cdots,\gamma_{l+1}}\sum_{i=1}^{l+1}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,l])}\left(\prod_{j=1}^{l}v^{\gamma_j\gamma_{j+1}}\right)\delta_{\gamma\gamma_1}\delta_{\gamma_{l+1}\alpha}. \tag{92}$$

Obviously, we have seen that it is valid for $l=1,2$. Suppose it is also valid for a given n, thus form eqs.(84) and $b_\gamma^{(l)}(0)=0,\ l\geq 1$ it follows that

$$b_\beta^{(n+1)}(t)=-\mathrm{i}\int_0^t \mathrm{d}\tau\sum_\gamma \mathrm{e}^{\mathrm{i}(E_\beta-E_\gamma)\tau}v^{\beta\gamma}b_\gamma^{(n)}(\tau). \tag{93}$$

Substitute eqs.(80) and (92) we obtain

$$\begin{aligned}
b_\beta^{(n+1)}(t) &= -\mathrm{i}\int_0^t \mathrm{d}\tau\sum_\gamma \mathrm{e}^{\mathrm{i}E_\beta\tau}v^{\beta\gamma}c_\gamma^{(n)}(\tau)\\
&= -\mathrm{i}\int_0^t \mathrm{d}\tau\sum_\gamma \mathrm{e}^{\mathrm{i}E_\beta\tau}v^{\beta\gamma}\sum_{\gamma_1,\cdots,\gamma_{n+1}}\sum_{i=1}^{n+1}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}\tau}}{d_i(E[\gamma,n])}\left(\prod_{j=1}^{n}v^{\gamma_j\gamma_{j+1}}\right)\delta_{\gamma\gamma_1}\delta_{\gamma_{n+1}\alpha}\\
&= \sum_{\gamma_1,\cdots,\gamma_{n+2}}\sum_{i=1}^{n+1}(-1)^{i-1}\frac{(\mathrm{e}^{-\mathrm{i}(E_{\gamma_i}-E_{\gamma_{n+2}})t}-1)}{d_i(E[\gamma,n])(E_{\gamma_i}-E_{\gamma_{n+2}})}\left(\prod_{j=1}^{n}v^{\gamma_j\gamma_{j+1}}\right)v^{\gamma_{n+2}\gamma_1}\delta_{\gamma_{n+2}\beta}\delta_{\gamma_{n+1}\alpha}.
\end{aligned} \tag{94}$$

In the last equality, we have inserted $\sum_{\gamma_{n+2}}\delta_{\gamma_{n+2}\beta}$, summed over γ and integral over τ. In terms of the definition of $d_i(E[\gamma,n]$, we know that $d_i(E[\gamma,n])(E_{\gamma_i}-E_{\gamma_{n+2}})=d_i(E[\gamma,n+1])$. Then based on our identity (26), we have

$$\sum_{i=1}^{n+1}(-1)^{i-1}\frac{1}{d_i(E[\gamma,n+1])}=-(-1)^{(n+2)-1}\frac{1}{d_{n+2}(E[\gamma,n+1])}. \tag{95}$$

Thus eq.(94) becomes

$$b_\beta^{(n+1)}(t) = \sum_{\gamma_1,\cdots,\gamma_{n+2}}\sum_{i=1}^{n+2}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,n+1])}\left(\prod_{j=1}^{n}v^{\gamma_j\gamma_{j+1}}\right)v^{\gamma_{n+2}\gamma_1}\delta_{\gamma_{n+2}\beta}\delta_{\gamma_{n+1}\alpha}\mathrm{e}^{\mathrm{i}E_\beta t}. \tag{96}$$

Set the index taking turns, that is, $\gamma_1\to\gamma_{n+2}$ and $\gamma_i+1\to\gamma_i,(n+1)\geq i\geq 1)$. Again from the definition of $d_i(E[\gamma,n]$, we can verify easily under above index taking turns,

$$\sum_{i=1}^{n+2}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,n+1])}\longrightarrow\sum_{i=1}^{n+2}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,n+1])}. \tag{97}$$

Therefore we obtain the conclusion

$$c_\beta^{(n+1)}(t)=b_\beta^{(n+1)}(t)\mathrm{e}^{-\mathrm{i}E_\beta t}=\sum_{\gamma_1,\cdots,\gamma_{n+2}}\sum_{i=1}^{n+2}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,n+1])}\left(\prod_{j=1}^{n+1}v^{\gamma_j\gamma_{j+1}}\right)\delta_{\beta\gamma_1}\delta_{\gamma_{n+2}\alpha}. \tag{98}$$

This implies that we finish the proof of expression (92). Substitute it into eq.(77), we immediately obtain the same expression as the our solution (35).

From the statement above, it can be seen that our solution actually finish the task to solve the recurrence equation of $b_\gamma^l(t)$ by using our new method. Although the recurrence equation of $b_\gamma^l(t)$ can be solved by integral in principle, it is indeed not easy if one does not to know our identity (26) and relevant relations. At our knowledge, the general term form has not ever been obtained up to now. This is an obvious difficult to attempt including high order approximations. However, our solution has done it, and so it will make some calculations of perturbation theory more convenient and more accurate.

However, we pay for the price to require that H is not explicitly dependent on time. Obviously our solution is written as a symmetric and whole form. Moreover, our method gives up, at least in the sense of formalization, the requirement that V is a small enough compared wit H_0, and our proof is more general and more strict. In addition, we do not need to consider the cases that the perturbing potential is switched at the initial and final time, but the perturbing potential is not explicitly dependent of time. Since we obtain the general term, our solution must have more applications, it is more efficient and more accurate for the practical applications, because we can selectively include partial contributions from high order even all order approximation. This can be called as the improved forms of perturbed solutions, which will be given in Sec. VII.

VI. TWO SKILLS IN THE IMPROVED SCHEME OF PERTURBATION THEORY

For the practical applications, we indeed can take the finite order approximations, that is, a finite l is given in our solution (35). Moreover, in order to show the advantages of our exact solution and provide the improved scheme of perturbation theory, we will derive out the improved forms of perturbative solution including the partial contributions from the higher order approximations. Because all of steps are well-regulated and only technology is to find the limitation of primary functions. In other words, the calculation of our exact solution and perturbation theory is operational in the practical applications, and further it can advance the efficiency and precision in the calculations. Frankly speaking, before we know our exact solution, we are puzzled by too many irregular terms and very trouble dependence on previous calculation steps. Moreover, we are often anxious about the result precision in such some calculations because those terms proportional to $t^a \mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}$ $(a=1,2,\cdots)$ in the high order approximation might not be ignorable with time increasing. Considering the contributions these terms can obviously improve the precision. However, the known perturbation theory does not give the general term, considering this task to add reasonably the high order approximations is actually impossible.

Since our exact solution has given the explicit form of any order approximation, that is a general term of an arbitrary order perturbed solution, and their forms are simply the summations of a series. Just enlightened by this general term of arbitrary order perturbed solution, we use two skills to develop the perturbation theory, which are respectively expressed in the following two subsections.

A. Redivision of Hamiltonian

The first skill is to decompose the matrix elements of H_1 in the representation of H_0 into diagonal part and nondiagonal part:

$$H_1^{\gamma_j\gamma_{j+1}} = h_1^{\gamma_j}\delta_{\gamma_j\gamma_{j+1}} + g_1^{\gamma_j\gamma_{j+1}}, \tag{99}$$

so that the concrete expression of a given order approximation can be easily calculated. Note that $h_1^{\gamma_j}$ has been chosen as its diagonal elements and then $g_1^{\gamma_j\gamma_{j+1}}$ has been set as its nondiagonal elements:

$$g_1^{\gamma_j\gamma_{j+1}} = g_1^{\gamma_j\gamma_{j+1}}(1-\delta_{\gamma_j\gamma_{j+1}}). \tag{100}$$

As examples, for the first order approximation, it is easy to calculate out that

$$A_1^{\gamma\gamma'}(h) = \sum_{\gamma_1,\gamma_2}\left[\sum_{i=1}^{2}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,l])}\right]\left(h_1^{\gamma_1}\delta_{\gamma_1\gamma_2}\right)\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_2} = \frac{(-\mathrm{i}h_1^{\gamma}t)}{1!}\mathrm{e}^{-\mathrm{i}E_{\gamma}t}\delta_{\gamma\gamma'}, \tag{101}$$

$$\begin{aligned} A_1^{\gamma\gamma'}(g) &= \sum_{\gamma_1,\gamma_2}\left[\sum_{i=1}^{2}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,l])}\right] g_1^{\gamma_1}(1-\delta_{\gamma_1\gamma_2})\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_2} \\ &= \left[\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{E_\gamma - E_{\gamma'}} - \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{E_\gamma - E_{\gamma'}}\right](1-\delta_{\gamma\gamma'}). \end{aligned} \tag{102}$$

Note that here and after we use the symbol $A_i^{\gamma\gamma'}$ denoting the contribution from ith order approximation, which is defined by (36), while its argument indicates the product form of matrix elements of H_1 in H_0 representation. However, for the second order approximation, since

$$\prod_{j=1}^{2} H_1^{\gamma_j\gamma_{j+1}} = \left(h_1^{\gamma_1}\right)^2\delta_{\gamma_1\gamma_2}\delta_{\gamma_2\gamma_3} + h_1^{\gamma_1}g_1^{\gamma_2\gamma_3}\delta_{\gamma_1\gamma_2} + g_1^{\gamma_1\gamma_2}h_1^{\gamma_2}\delta_{\gamma_2\gamma_3} + g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma_3}. \tag{103}$$

we need to calculate the mixed product of diagonal and nondiagonal elements of H_1. Obviously, we have

$$A_2^{\gamma\gamma'}(hh) = \sum_{\gamma_1,\gamma_2,\gamma_3}\left[\sum_{i=1}^{3}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,2])}\right]\left(h_1^{\gamma_1}\delta_{\gamma_1\gamma_2}h_1^{\gamma_2}\delta_{\gamma_2\gamma_3}\right)\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_3} = \frac{(-\mathrm{i}h_1^{\gamma}t)^2}{2!}\mathrm{e}^{-\mathrm{i}E_{\gamma}t}\delta_{\gamma\gamma'}, \tag{104}$$

$$A_2^{\gamma\gamma'}(hg) = \sum_{\gamma_1,\gamma_2,\gamma_3}\left[\sum_{i=1}^{3}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,2])}\right]h_1^{\gamma_1}g_1^{\gamma_2\gamma_3}\delta_{\gamma_1\gamma_2}\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_3}$$
$$= \left[-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma'})^2}+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2}+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{E_\gamma-E_{\gamma'}}\right]h_1^{\gamma}g_1^{\gamma\gamma'}, \tag{105}$$

$$A_2^{\gamma\gamma'}(gh) = \sum_{\gamma_1,\gamma_2,\gamma_3}\left[\sum_{i=1}^{3}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,2])}\right]h_1^{\gamma_2}g_1^{\gamma_1\gamma_2}\delta_{\gamma_2\gamma_3}\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_3}$$
$$= \left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma'})^2}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2}-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{E_\gamma-E_{\gamma'}}\right]g_1^{\gamma\gamma'}h_1^{\gamma'}, \tag{106}$$

$$A_2^{\gamma\gamma'}(gg) = \sum_{\gamma_1,\gamma_2,\gamma_3}\left[\sum_{i=1}^{3}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,2])}\right]g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma_3}\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_3}$$
$$= \sum_{\gamma_1}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma'})}\right.$$
$$\left.+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})}\right]g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}. \tag{107}$$

In the usual time-dependent perturbation theory, the zeroth order approximation of time evolution of quantum state keeps its original form

$$\left|\Psi^{(0)}(t)\right\rangle = \mathrm{e}^{-\mathrm{i}E_\gamma t}|\Phi^\gamma\rangle, \tag{108}$$

where we have set the initial state as $|\Phi^\gamma\rangle$ for simplicity. By using our solution, we easily calculate out the contributions of all of order approximations from the product of completely diagonal elements h to this zeroth order approximation

$$\sum_{\gamma_1,\cdots,\gamma_{l+1}}\left[\sum_{i=1}^{l+1}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,l])}\right]\left(\prod_{j=1}^{l}h_1^{\gamma_j}\delta_{\gamma_j\gamma_{j+1}}\right)\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_{l+1}}$$
$$= \frac{(-\mathrm{i}h_1^{\gamma}t)^l}{l!}\mathrm{e}^{-\mathrm{i}E_\gamma t}\delta_{\gamma\gamma'}. \tag{109}$$

Therefore, we can add the contributions of all of order approximation parts from the product of completely diagonal elements h to this zeroth order approximation to obtain

$$\left|\Psi'^{(0)}(t)\right\rangle = \mathrm{e}^{-\mathrm{i}\left(E_\gamma+h_1^\gamma\right)t}|\Phi^\gamma\rangle. \tag{110}$$

Similarly, by calculation, we can deduce out that up to the second approximation, the perturbed solution has the following form

$$|\Psi'(t)\rangle = \sum_{\gamma,\gamma'}\left\{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+h_1^\gamma\right)t}\delta_{\gamma\gamma'}+\left[\frac{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+h_1^\gamma\right)t}-\mathrm{e}^{-\mathrm{i}\left(E_{\gamma'}+h_1^{\gamma'}\right)t}}{(E_\gamma+h_1^\gamma)-\left(E_{\gamma'}+h_1^{\gamma'}\right)}\right]g_1^{\gamma\gamma'}\right.$$
$$+\sum_{\gamma_1}\left[\frac{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+h_1^\gamma\right)t}}{[(E_\gamma+h_1^\gamma)-(E_{\gamma_1}+h_1^{\gamma_1})]\left[(E_\gamma+h_1^\gamma)-\left(E_{\gamma'}+h_1^{\gamma'}\right)\right]}\right.$$
$$-\frac{\mathrm{e}^{-\mathrm{i}\left(E_{\gamma_1}+h_1^{\gamma_1}\right)t}}{[(E_\gamma+h_1^\gamma)-(E_{\gamma_1}+h_1^{\gamma_1})]\left[(E_{\gamma_1}+h_1{}_1^{\gamma})-\left(E_{\gamma'}+h_1^{\gamma'}\right)\right]}$$
$$\left.\left.+\frac{\mathrm{e}^{-\mathrm{i}\left(E_{\gamma'}+h_1^{\gamma'}\right)t}}{\left[(E_\gamma+h_1^\gamma)-\left(E_{\gamma'}+h_1^{\gamma'}\right)\right]\left[(E_{\gamma_1}+h_1^{\gamma_1})-\left(E_{\gamma'}+h_1^{\gamma'}\right)\right]}\right]g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\right\}$$
$$\left[\langle\Phi^{\gamma'}|\Psi(0)\rangle\right]|\Phi^\gamma\rangle+\mathcal{O}(H_1^3). \tag{111}$$

However, for the higher order approximation, the corresponding calculation is heavy. In fact, it is unnecessary to calculate the contributions from those terms with the diagonal elements of H_1 since introducing the following skill. This is a reason why we omit the relevant calculation details. Here we mention it only for verifying the correctness of our exact solution in this way.

The results (110) and (111) are not surprised because of the fact that the Hamiltonian is re-divisible. Actually, we can furthermore use a trick of redivision of the Hamiltonian so that the new H_0 contains the diagonal part of H_1, that is

$$H_0' = H_0 + \sum_{\gamma} h_1^{\gamma} |\Phi^{\gamma}\rangle\langle\Phi^{\gamma}|, \tag{112}$$

$$H_1' = H_1 - \sum_{\gamma} h_1^{\gamma} |\Phi^{\gamma}\rangle\langle\Phi^{\gamma}| = \sum_{\gamma,\gamma'} g_1^{\gamma\gamma'} |\Phi^{\gamma}\rangle\langle\Phi^{\gamma'}|. \tag{113}$$

In other words, without loss of generality, we always can choose that H_1' has only the nondiagonal elements in the H_0' (or H_0) representation and

$$H_0'|\Phi^{\gamma}\rangle = \left(E_{\gamma} + h_1^{\gamma}\right)|\Phi^{\gamma}\rangle = E_{\gamma}'|\Phi^{\gamma}\rangle. \tag{114}$$

In spite of our skill is so simple, it seems not be sufficiently transpired and understood from the fact that the recent some textbooks of quantum mechanics still remain the diagonal contribution from the unperturbed part in the expression of the second order perturbed state.

It must be emphasized that the skill to redivide Hamiltonian leads to the fact that the new perturbed solution can be obtained by the replacement

$$E_{\gamma_i} \rightarrow E_{\gamma_i} + h_1^{\gamma_i} \tag{115}$$

in the usual perturbed solution and its conclusions. This implies that there are two equivalent ways to obtain the same perturbed solution and its conclusions. One of them is to redefine the energy level E_{γ_i} as E_{γ_i}', think E_{γ_i}' to be explicitly independent on the perturbed parameter from a redefined view, and then use the method in the usual perturbation theory to obtain the result from the redivided H_1'. The other way is directly deduce out the perturbed solution from the original Hamiltonian by using the standard procedure, but a rearrangement and summation are carried out just like above done by us. From mathematical view, this is because the perturbed parameter is only formal multiplier and it is introduced after redefining E_{γ_i}'. The first way or technology will be again applied to our scheme to obtain improved forms of perturbed energy and perturbed state in Sec. IX.

For simplicity, in the following, we omit the $'$ in H_0, H_1 as well as E_γ, and always let H_1 have only its nondiagonal part unless particular claiming.

B. Contraction and anti-contraction of nondiagonal element product

In this subsection, we present the second important skill enlightened by our exact solution, this is, a method to calculate the contributions from the contractions and anti-contractions of nondiagonal element product in a given order approximation. This method is also the most important technology in our scheme of perturbation theory.

Let us start with the second order approximation. Since we have taken $H_1^{\gamma_j\gamma_{j+1}}$ only with the nondiagonal part $g_1^{\gamma_j\gamma_{j+1}}$, the contribution from the second order approximation is only $A_2^{\gamma\gamma'}(gg)$ in eq.(107). However, we find that the limitation in the expression of $A_2^{\gamma\gamma'}(gg)$ has not been completely found out because we have not excluded the case $E_\gamma = E_{\gamma'}$ (or $\gamma = \gamma'$). This problem can be fixed by introducing a decomposition

$$g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} = g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} \delta_{\gamma_1\gamma_3} + g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} \eta_{\gamma_1\gamma_3}, \tag{116}$$

where $\eta_{\gamma_1\gamma_3} = 1 - \delta_{\gamma_1\gamma_3}$. Thus, the contribution from the second order approximation is made of two terms, one so-called contraction term with the δ function and another so-called anti-contraction term with the η function. It must be emphasized that we only consider the non-degenerate case here and after for simplification. While when the degeneration happens, two indexes with the same main energy level number will not have the anti-contraction.

In terms of above skill, we find that the contribution from the second order approximation is made of the corresponding contraction- and anti-contraction- terms

$$A_2^{\gamma\gamma'}(gg) = A_2^{\gamma\gamma'}(gg;c) + A_2^{\gamma\gamma'}(gg;n), \tag{117}$$

where

$$
\begin{aligned}
A_2^{\gamma\gamma'}(gg;c) &= \sum_{\gamma_1,\gamma_2,\gamma_3}\left[\sum_{i=1}^{3}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,2])}\right] g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3}\delta_{\gamma_1\gamma_3}\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_3} \\
&= \sum_{\gamma_1}\left[-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma-E_{\gamma_1})^2}+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2}+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{E_\gamma-E_{\gamma_1}}\right]\left|g_1^{\gamma\gamma_1}\right|^2\delta_{\gamma\gamma'},
\end{aligned}
\tag{118}
$$

$$
\begin{aligned}
A_2^{\gamma\gamma'}(gg;n) &= \sum_{\gamma_1,\gamma_2,\gamma_3}\left[\sum_{i=1}^{3}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,2])}\right] g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3}\eta_{\gamma_1\gamma_3}\delta_{\gamma\gamma_1}\delta_{\gamma'\gamma_3} \\
&= \sum_{\gamma_1}\left[\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma'})}\right. \\
&\quad \left.+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})}\right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}.
\end{aligned}
\tag{119}
$$

This method can be extended to the higher order approximation by introducing a concept of g-product decomposition. For a sequential product of nondiagonal elements g with the form $\prod_{k=1}^{m} g_1^{\gamma_k\gamma_{k+1}}$ $(m \geq 2)$, we define its $(m-1)$th decomposition by

$$
\prod_{k=1}^{m} g_1^{\gamma_k\gamma_{k+1}} = \left(\prod_{k=1}^{m} g_1^{\gamma_k\gamma_{k+1}}\right)\delta_{\gamma_1\gamma_{m+1}} + \left(\prod_{k=1}^{m} g_1^{\gamma_k\gamma_{k+1}}\right)\eta_{\gamma_1\gamma_{m+1}}.
\tag{120}
$$

When we calculate the contributions from the nth order approximation, we first will carry out $n-1$ the first decompositions, that is

$$
\prod_{k=1}^{n} g_1^{\gamma_k\gamma_{k+1}} = \left(\prod_{k=1}^{n} g_1^{\gamma_k\gamma_{k+1}}\right)\left[\prod_{k=1}^{n-1}\left(\delta_{\gamma_k\gamma_{k+2}}+\eta_{\gamma_k\gamma_{k+2}}\right)\right].
\tag{121}
$$

Obviously, from the fact that H_1 is usually taken as Hermit one, it follows that

$$
g_1^{\gamma_j\gamma_{j+1}} g_1^{\gamma_{j+1}\gamma_{j+2}}\delta_{\gamma_j\gamma_{j+2}} = \left|g_1^{\gamma_j\gamma_{j+1}}\right|^2\delta_{\gamma_j\gamma_{j+2}}.
\tag{122}
$$

When considering the contributions a given order approximation, the summation over one of two subscripts will lead in the contraction of g-production. More generally, for the contraction of even number g-production

$$
\left(\prod_{j=1}^{m} g_1^{\gamma_j\gamma_{j+1}}\prod_{k=1}^{m-1}\delta_{\gamma_k\gamma_{k+2}}\right)\delta_{\gamma_1\gamma}\delta_{\gamma_{m+1}\gamma'} = \left|g_1^{\gamma\gamma_2}\right|^m\left(\prod_{k=1}^{m-1}\delta_{\gamma_k\gamma_{k+2}}\right)\delta_{\gamma_1\gamma}\delta_{\gamma_{m+1}\gamma'}\delta_{\gamma\gamma'},
\tag{123}
$$

and for the contraction of odd number g-production,

$$
\left(\prod_{j=1}^{m} g_1^{\gamma_j\gamma_{j+1}}\prod_{k=1}^{m-1}\delta_{\gamma_k\gamma_{k+2}}\right)\delta_{\gamma_1\gamma}\delta_{\gamma_{m+1}\gamma'} = \left|g_1^{\gamma\gamma'}\right|^{m-1}\left(\prod_{k=1}^{m-1}\delta_{\gamma_k\gamma_{k+2}}\right)\delta_{\gamma_1\gamma}\delta_{\gamma_{m+1}\gamma'} g_1^{\gamma\gamma'},
\tag{124}
$$

where $\delta_{\gamma_1\gamma}\delta_{\gamma_{m+1}}$ is a factor appearing in the expression of our solution.

Then, we consider, in turn, all possible the second decompositions, the third decompositions, and up to the $(n-1)$th decompositions. It must be emphasized that after calculating the contributions from terms of lower decompositions, some of terms in the higher decompositions may be trivial because there are some symmetric and complementary symmetric indexes in the corresponding result, that is, the product of this result and the given $\delta_{\gamma_k\gamma_{k'}}$ or $\eta_{\gamma_k\gamma_{k'}}$ is zero. In other words, such some higher decompositions do not need to be considered. As an example, let us analysis the contribution from the third order approximation. It is clear that the first decomposition of a sequential production of three nondiagonal elements becomes

$$
\begin{aligned}
g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4} &= g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4}\delta_{\gamma_1\gamma_3}\delta_{\gamma_2\gamma_4} + g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4}\delta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4} \\
&\quad + g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4}\eta_{\gamma_1\gamma_3}\delta_{\gamma_2\gamma_4} + g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4}\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4}.
\end{aligned}
\tag{125}
$$

Thus, the related contribution is just divided as 4 terms

$$A_3^{\gamma\gamma'}(ggg) = A_3^{\gamma\gamma'}(ggg;cc) + A_3^{\gamma\gamma'}(ggg;cn) + A_3^{\gamma\gamma'}(ggg;nc) + A_3^{\gamma\gamma'}(ggg,nn). \tag{126}$$

In fact, by calculating we know that the second decomposition of the former three terms do not need to be considered, only the second decomposition of the last term is nontrivial. This means that

$$A_3^{\gamma\gamma'}(ggg;nn) = A_3^{\gamma\gamma'}(ggg;nn,c) + A_3^{\gamma\gamma'}(ggg;nn,n), \tag{127}$$

where we have added $\delta_{\gamma_1\gamma_3}$ in the definition of $A_3^{\gamma\gamma'}(ggg;nn,c)$, and $\eta_{\gamma_1\gamma_3}$ in the definition of $A_3^{\gamma\gamma'}(ggg;nn,n)$. Obviously, in the practical process, this feature largely simplifies the calculations. It is easy to see that the number of all of terms with contractions and anti-contractions is 5. For convenience and clearness, we call the contributions from the different terms in the decomposition of g-product as the contractions and anti-contractions of g-product. Of course, the contraction and anti-contraction refer to the meaning after summation(s) over the subscript(s) in general. Moreover, here and after, we first drop the argument $gg\cdots g$ in the ith order approximation A_i since its meaning has been indicated by i after the Hamiltonian is redivided. For example, the explicit expressions of all contraction- and anti-contraction terms in the third order approximation A_3 can be calculated as the following:

$$\begin{aligned}
A_3^{\gamma\gamma'}(cc) &= \sum_{\gamma_1,\cdots,\gamma_4}\left[\sum_{i=1}^{4}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,3])}\right]\left[\prod_{j=1}^{3}g_1^{\gamma_j\gamma_{j+1}}\right]\left(\prod_{k=1}^{2}\delta_{\gamma_k\gamma_{k+2}}\right)\delta_{\gamma_1\gamma}\delta_{\gamma_4\gamma'} \\
&= \left[-\frac{2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma'})^3}+\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^3}+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma'})^2}\right. \\
&\quad\left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2}\right]\left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma'},
\end{aligned} \tag{128}$$

$$\begin{aligned}
A_3^{\gamma\gamma'}(cn) &= \sum_{\gamma_1,\cdots,\gamma_{l+1}}\left[\sum_{i=1}^{4}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,3])}\right]\left[\prod_{j=1}^{3}g_1^{\gamma_j\gamma_{j+1}}\right]\delta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4}\delta_{\gamma_1\gamma}\delta_{\gamma_{l+1}\gamma'} \\
&= \sum_{\gamma_1}\left[-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})^2}-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma'})}\right. \\
&\quad+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma'})}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})} \\
&\quad\left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})}\right]\left|g_1^{\gamma\gamma_1}\right|^2 g_1^{\gamma\gamma'}\eta_{\gamma_1\gamma'},
\end{aligned} \tag{129}$$

$$\begin{aligned}
A_3^{\gamma\gamma'}(nc) &= \sum_{\gamma_1,\cdots,\gamma_4}\left[\sum_{i=1}^{4}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,3])}\right]\left[\prod_{j=1}^{3}g_1^{\gamma_j\gamma_{j+1}}\right]\eta_{\gamma_1\gamma_3}\delta_{\gamma_2\gamma_4}\delta_{\gamma_1\gamma}\delta_{\gamma_{l+1}\gamma'} \\
&= \sum_{\gamma_1}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})^2}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma'})^2}\right. \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})^2}+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma'})^2} \\
&\quad\left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})}\right]g_1^{\gamma\gamma'}\left|g_1^{\gamma_1\gamma'}\right|^2\eta_{\gamma\gamma_1},
\end{aligned} \tag{130}$$

$$
\begin{aligned}
A_3^{\gamma\gamma'}(nn,c) &= \sum_{\gamma_1,\cdots,\gamma_4}\left[\sum_{i=1}^{4}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,3])}\right]\left[\prod_{j=1}^{3}g_1^{\gamma_j\gamma_{j+1}}\right]\delta_{\gamma_1\gamma}\delta_{\gamma_4\gamma'}\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4}\delta_{\gamma\gamma'}\\
&= \sum_{\gamma_1\gamma_2}\left[-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_{\gamma_1}-E_{\gamma_2}\right)}+\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_2}\right)^2\left(E_{\gamma_1}-E_{\gamma_2}\right)}\right.\\
&\quad+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_{\gamma_1}-E_{\gamma_2}\right)}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{\left(E_\gamma-E_{\gamma_2}\right)^2\left(E_{\gamma_1}-E_{\gamma_2}\right)}\\
&\quad\left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_{\gamma_1}-E_{\gamma_2}\right)}\right]g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_2}\delta_{\gamma\gamma'},
\end{aligned}
\tag{131}
$$

$$
\begin{aligned}
A_3^{\gamma\gamma'}(nn,n) &= \sum_{\gamma_1,\cdots,\gamma_4}\left[\sum_{i=1}^{4}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,3])}\right]\left[\prod_{j=1}^{3}g_1^{\gamma_j\gamma_{j+1}}\right]\delta_{\gamma_1\gamma}\delta_{\gamma_4\gamma'}\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4}\eta_{\gamma\gamma'}\\
&\quad+\sum_{\gamma_1\gamma_2}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma_2}\right)\left(E_\gamma-E_{\gamma'}\right)}\right.\\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_{\gamma_1}-E_{\gamma_2}\right)\left(E_{\gamma_1}-E_{\gamma'}\right)}\\
&\quad+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{\left(E_\gamma-E_{\gamma_2}\right)\left(E_{\gamma_1}-E_{\gamma_2}\right)\left(E_{\gamma_2}-E_{\gamma'}\right)}\\
&\quad\left.-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{\left(E_\gamma-E_{\gamma'}\right)\left(E_{\gamma_1}-E_{\gamma'}\right)\left(E_{\gamma_2}-E_{\gamma'}\right)}\right]g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma'}\eta_{\gamma\gamma'}.
\end{aligned}
\tag{132}
$$

In above calculations, the used technologies are mainly to find the limitation, dummy index changing and summation, as well as the replacement $g_1^{\gamma_i\gamma_j}\eta_{\gamma_i\gamma_j}=g_1^{\gamma_i\gamma_j}$ since $g_1^{\gamma_i\gamma_j}$ has been nondiagonal.

It must be emphasized that, in our notation, $A_i^{\gamma\gamma'}$ represents the contributions from the ith order approximation. The other independent variables are divided into $i-1$ groups and are arranged sequentially corresponding to the order of g-product decomposition. That is, the first variable group represents the first decompositions, the second variable group represents the second decompositions, and so on. Every variable group is a bit-string made of three possible element c,n,k and its length is equal to the number of the related order of g-product decomposition, that is, for the jth decompositions in the ith order approximation its length is $i-j$. In each variable group, c corresponds to a δ function, n corresponds to a η function and k corresponds to 1 (non-decomposition). Their sequence in the bit-string corresponds to the sequence of contraction and/or anti-contraction index string. From the above analysis and statement, the index string of the jth decompositions in the i order approximation is:

$$
\prod_{k=1}^{i-j}\left(\gamma_k,\gamma_{k+1+j}\right). \tag{133}
$$

For example, the first variable group is $cccn$, which refers to the first decomposition in five order approximation and the contribution term to include the factor $\delta_{\gamma_1\gamma_3}\delta_{\gamma_2\gamma_4}\delta_{\gamma_3\gamma_5}\eta_{\gamma_4\gamma_6}$ in the definition of $A_5(cccn)$. Similarly, $cncc$ means to insert the factor $\delta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4}\delta_{\gamma_3\gamma_5}\delta_{\gamma_4\gamma_6}$ in the definition of $A_5(cncc)$. When there are non trivial second contractions, for instance, two variable group $(ccnn,kkc)$ represents that the definition of $A_5(ccnn,kkc)$ has the factor $\left(\delta_{\gamma_1\gamma_3}\delta_{\gamma_2\gamma_4}\eta_{\gamma_3\gamma_5}\eta_{\gamma_4\gamma_6}\right)\delta_{\gamma_3\gamma_6}$. Since there are fully trivial contraction (the bit-string is made of only k), we omit their related variable group for simplicity.

Furthermore, we pack up all the contraction- and non-contraction terms in the following way so that we can obtain conveniently the improved forms of perturbed solution of dynamics including the partial contributions from the high order approximation. We first decompose $A_3^{\gamma\gamma'}$, which is a summation of all above terms, into the three parts according to $\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}$, $(-\mathrm{i}t)\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}$ and $(-\mathrm{i}t)^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}/2$:

$$
A_3^{\gamma\gamma'}=A_3^{\gamma\gamma'}(\mathrm{e})+A_3^{\gamma\gamma'}(t\mathrm{e})+A_3^{\gamma\gamma'}(t^2\mathrm{e}). \tag{134}
$$

Secondly, we decompose its every term into three parts according to $\mathrm{e}^{-\mathrm{i}E_\gamma t}, \mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}$ $(\sum_{\gamma_1} \mathrm{e}^{-\mathrm{i}E_{\gamma_1} t})$ and $\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}$:

$$A_3^{\gamma\gamma'}(\mathrm{e}) = A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_\gamma t}) + A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}) + A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}), \tag{135}$$

$$A_3^{\gamma\gamma'}(t\mathrm{e}) = A_3^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_\gamma t}) + A_3^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}) + A_3^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}), \tag{136}$$

$$A_4^{\gamma\gamma'}(t^2\mathrm{e}) = A_3^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t}) + A_3^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}) + A_3^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}). \tag{137}$$

Finally, we again decompose every term in above equations into the diagonal and non-diagonal parts about γ and γ':

$$A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t}) = A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{D}) + A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{N}), \tag{138}$$

$$A_3^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t}) = A_3^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{D}) + A_3^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{N}), \tag{139}$$

$$A_3^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t}) = A_3^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{D}) + A_3^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{N}), \tag{140}$$

where E_{γ_i} takes E_γ, E_{γ_1} and $E_{\gamma'}$.

According above way, it is easy to obtain

$$A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_\gamma t};\mathrm{D}) = -\sum_{\gamma_1,\gamma_2} \mathrm{e}^{-\mathrm{i}E_\gamma t}\left[\frac{1}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_1})^2} + \frac{1}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_1})}\right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'}, \tag{141}$$

$$\begin{aligned}A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_\gamma t};\mathrm{N}) = & -\sum_{\gamma_1} \mathrm{e}^{-\mathrm{i}E_\gamma t}\left[\frac{1}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^2} + \frac{1}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma'})}\right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'} \\ & + \sum_{\gamma_1,\gamma_2} \mathrm{e}^{-\mathrm{i}E_\gamma t}\frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})},\end{aligned} \tag{142}$$

$$A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t};\mathrm{D}) = \sum_{\gamma_1,\gamma_2} \mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}\frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2(E_{\gamma_1} - E_{\gamma_2})}, \tag{143}$$

$$A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t};\mathrm{N}) = -\sum_{\gamma_1,\gamma_2} \mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}\frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'}\eta_{\gamma_1\gamma'}\eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})}, \tag{144}$$

$$A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t};\mathrm{D}) = -\sum_{\gamma_1,\gamma_2} \mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}\frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_2})^2(E_{\gamma_1} - E_{\gamma_2})}, \tag{145}$$

$$A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t};\mathrm{N}) = \sum_{\gamma_1,\gamma_2} \mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}\frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma'})}, \tag{146}$$

$$A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma'} t};\mathrm{D}) = 0, \tag{147}$$

$$\begin{aligned}A_3^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma'} t};\mathrm{N}) = & \sum_{\gamma_1} \mathrm{e}^{-\mathrm{i}E_{\gamma'} t}\left[\frac{1}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})^2} + \frac{1}{(E_\gamma - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})}\right] g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma'} g_1^{\gamma\gamma'} \\ & - \sum_{\gamma_1,\gamma_2} \mathrm{e}^{-\mathrm{i}E_{\gamma'} t}\frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'}\eta_{\gamma_1\gamma'}\eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})}.\end{aligned} \tag{148}$$

In the end of this subsection, we would like to point out that one of purposes introducing the decompositions of g-product and the contractions and anti-contractions of g-product is to eliminate the apparent (not real) singularities and find out all the limitations from the contributions of g-product contractions. It is important to express the results with the physical significance.

VII. IMPROVED FORMS OF PERTURBED SOLUTION OF DYNAMICS

In fact, the final aim using the decomposition and then obtaining contraction and anti-contraction is to include partially contributions from the high order approximations of nondiagonal part of H_1 in our improved scheme of perturbation theory. In this section, making use of the skill to calculate contractions and anti-contractions of g-product, we can obtain the zeroth, first, second and third order improved forms of perturbed solutions including partial higher order approximations.

In mathematics, the process to obtain the improved forms of perturbed solutions is a kind of skill to deal with an infinity series, that is, according to some principles and the general term form to rearrange those terms with the same features together forming a group, then sum all of the terms in such a particular group that they become a compact function at a given precision, finally this infinity series is transformed into a new form that directly relates with the studied problem. More concretely speaking, since we concern the system developing with time t, we take those terms with $(-\mathrm{i}y_i t)\mathrm{e}^{-\mathrm{i}x_i t}$, $(-\mathrm{i}y_i t)^2\mathrm{e}^{-\mathrm{i}x_i t}/2!$ and $(-\mathrm{i}y_i t)^3\mathrm{e}^{-\mathrm{i}x_i t}/3!$, $\cdots$ with the same factor f together forming a group, then sum them to obtain an exponential function $f\exp\left[-\mathrm{i}\left(x_i+y_i\right)t\right]$. The physical reason to do this is such an exponential function represents the system evolution in theory and has the obvious physical significance in the calculation of transition probability and perturbed energy. Through rearranging and summing, those terms with factors $t^a\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}$, $(a=1,2,\cdots)$ in the higher order approximation are added to the improved lower approximations, we thus can advance the precision, particular, when the evolution time t is longer. We can call it "dynamical rearrangement and summation" technology.

A. Improved form of the zeroth order perturbed solution of dynamics

Let us start with the zeroth order perturbed solution of dynamics. In the usual perturbation theory, it is well-known

$$\left|\Psi^{(0)}(t)\right\rangle=\sum_{\gamma}\mathrm{e}^{-\mathrm{i}E_\gamma t}\langle\Phi^\gamma|\Psi(0)\rangle|\Phi^\gamma\rangle=\sum_{\gamma\gamma'}\mathrm{e}^{-\mathrm{i}E_\gamma t}\delta_{\gamma\gamma'}a_{\gamma'}|\Phi^\gamma\rangle, \tag{149}$$

where $a_{\gamma'}=\langle\Phi^{\gamma'}|\Psi(0)\rangle$. Now, we would like to improve it so that it can include the partial contributions from higher order approximations. Actually, we can find that $A_2(c)$ and $A_3(nn,c)$ have the terms proportional to $(-\mathrm{i}t)$

$$(-\mathrm{i}t)\mathrm{e}^{-\mathrm{i}E_\gamma t}\left[\sum_{\gamma_1}\frac{1}{E_\gamma-E_{\gamma_1}}\left|g_1^{\gamma\gamma_1}\right|^2\right]\delta_{\gamma\gamma'}, \tag{150}$$

$$(-\mathrm{i}t)\mathrm{e}^{-\mathrm{i}E_\gamma t}\left[\sum_{\gamma_1,\gamma_2}\frac{1}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\right]\delta_{\gamma\gamma'}. \tag{151}$$

Introduce the notation

$$G_\gamma^{(2)}=\sum_{\gamma_1}\frac{1}{E_\gamma-E_{\gamma_1}}\left|g_1^{\gamma\gamma_1}\right|^2, \tag{152}$$

$$G_\gamma^{(3)}=\sum_{\gamma_1,\gamma_2}\frac{1}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma}. \tag{153}$$

It is clear that $G_\gamma^{(a)}$ has energy dimension, and we will see that it can be called the ath revision energy. Let us add the terms (150), (151) and the related terms in $A_4(t\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{D}), A_4(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{D}), A_5(t\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{D})$, $A_5(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{D})$, $A_6(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{D})$ and $A_6(t^3\mathrm{e}^{-\mathrm{i}E_\gamma t})$ given in Appendix B together, that is,

$$\begin{aligned}A_{0;\mathrm{I}}^{\gamma\gamma'}&=\mathrm{e}^{-\mathrm{i}E_\gamma t}\left[1+(-\mathrm{i}t)\left(G_\gamma^{(2)}+G_\gamma^{(3)}+G_\gamma^{(4)}+G_\gamma^{(5)}\right)\right.\\&\quad\left.+\frac{(-\mathrm{i}t)^2}{2!}\left(G_\gamma^{(2)}+G_\gamma^{(3)}\right)^2+\frac{(-\mathrm{i}t)^2}{2!}2G_\gamma^{(2)}G_\gamma^{(4)}+\cdots\right]\delta_{\gamma\gamma'},\end{aligned} \tag{154}$$

Although we have not finished the more calculations, from the mathematical symmetry and physical concept, we can think

$$\begin{aligned}A_{0;\mathrm{I}}^{\gamma\gamma'}&=\mathrm{e}^{-\mathrm{i}E_\gamma t}\left[1+(-\mathrm{i}t)\left(G_\gamma^{(2)}+G_\gamma^{(3)}+G_\gamma^{(4)}+G_\gamma^{(5)}\right)\right.\\&\quad\left.+\frac{(-\mathrm{i}t)^2}{2!}\left(G_\gamma^{(2)}+G_\gamma^{(3)}+G_\gamma^{(4)}+G_\gamma^{(5)}\right)^2+\cdots\right]\delta_{\gamma\gamma'},\end{aligned} \tag{155}$$

New terms can appear at A_7, A_8, A_9 and A_{10}. So we obtain the improved form of zeroth order perturbed solution of dynamics

$$\left|\Psi_{E_T}^{(0)}(t)\right\rangle_{\mathrm{I}} = \sum_{\gamma\gamma'} \mathrm{e}^{-\mathrm{i}\left(E_\gamma+G_\gamma^{(2)}+G_\gamma^{(3)}+G_\gamma^{(4)}+G_\gamma^{(5)}\right)t}\delta_{\gamma\gamma'}a_{\gamma'}|\Phi^\gamma\rangle. \tag{156}$$

It is clear that $G_\gamma^{(2)}$ is real. In fact, $G_\gamma^{(3)}$ is also real. In order to prove it, we exchange the dummy indexes γ_1 and γ_2 and take the complex conjugate of $G_\gamma^{(3)}$, that is

$$\begin{aligned} G_\gamma^{(3)*} &= \sum_{\gamma_1,\gamma_2} \frac{1}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})}\left(g_1^{\gamma\gamma_2}\right)^*\left(g_1^{\gamma_2\gamma_1}\right)^*\left(g_1^{\gamma_1\gamma}\right)^* \\ &= \sum_{\gamma_1,\gamma_2} \frac{1}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})} g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma} \\ &= G_\gamma^{(3)}, \end{aligned} \tag{157}$$

where we have used the relations $\left(g_1^{\beta_1\beta_2}\right)^* = g_1^{\beta_2\beta_1}$ for any β_1 and β_2 since H_1 is Hermit. Similar analyses can be applied to $G_\gamma^{(4)}$ and $G_\gamma^{(5)}$. These mean that $\mathrm{e}^{-\mathrm{i}\left(G_\gamma^{(2)}+G_\gamma^{(3)}+G_\gamma^{(4)}+G_\gamma^{(5)}\right)t}$ is still an oscillatory factor.

B. Improved form of the first order perturbed solution of dynamics

Furthermore, in order to include the partial contributions from higher than zeroth order approximation, we need to consider the contributions from nondiagonal elements in the higher order approximations.

Well-known usual first order perturbed part of dynamics is

$$\left|\Psi^{(1)}(t)\right\rangle = \sum_{\gamma,\gamma'}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{E_\gamma-E_{\gamma'}}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{E_\gamma-E_{\gamma'}}\right]H_1^{\gamma\gamma'}|\Phi^\gamma\rangle = \sum_{\gamma,\gamma'}\left[\left(\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{E_\gamma-E_{\gamma'}}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{E_\gamma-E_{\gamma'}}\right)g_1^{\gamma\gamma'}\right]|\Phi^\gamma\rangle. \tag{158}$$

It must be emphasized that H_1 is taken as only with the nondiagonal part for simplicity. That is, we have used the skill one stated above.

Thus, from $A_3(t\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{N})$ and $A_4(t\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{N})$, $A_4(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{D})$, $A_5(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{N})$, $A_6(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{N})$ in the Appendix B, it follows that

$$\begin{aligned} A_{1;\mathrm{I}}^{\gamma\gamma'} &= \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma'})}\left[1+(-\mathrm{i}t)\left(G_\gamma^{(2)}+G_\gamma^{(3)}+G_\gamma^{(4)}\right)\right. \\ &\quad \left.+\frac{(-\mathrm{i}t)^2}{2!}\left(G_\gamma^{(2)}\right)^2+\frac{(-\mathrm{i}t)^2}{2!}2G_\gamma^{(2)}G_\gamma^{(3)}+\cdots\right]g_1^{\gamma\gamma'} \\ &\quad -\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})}\left[1+(-\mathrm{i}t)\left(G_{\gamma'}^{(2)}+G_{\gamma'}^{(3)}+G_{\gamma'}^{(4)}\right)\right. \\ &\quad \left.+\frac{(-\mathrm{i}t)^2}{2!}\left(G_{\gamma'}^{(2)}\right)^2+\frac{(-\mathrm{i}t)^2}{2!}2G_{\gamma'}^{(2)}G_{\gamma'}^{(3)}+\cdots\right]g_1^{\gamma\gamma'}. \end{aligned} \tag{159}$$

Therefore, the improved perturbed solution of dynamics is just

$$\left|\Psi^{(1)}(t)\right\rangle_{\mathrm{I}} = \sum_{\gamma,\gamma'}\left[\left(\frac{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+G_\gamma^{(2)}+G_\gamma^{(3)}+G_\gamma^{(4)}\right)t}}{E_\gamma-E_{\gamma'}}-\frac{\mathrm{e}^{-\mathrm{i}\left(E_{\gamma'}+G_{\gamma'}^{(2)}+G_{\gamma'}^{(3)}+G_{\gamma'}^{(4)}\right)t}}{E_\gamma-E_{\gamma'}}\right)g_1^{\gamma\gamma'}\right]a_{\gamma'}|\Phi^\gamma\rangle. \tag{160}$$

C. Improved second order- and third order perturbed solution

Likewise, it is not difficult to obtain

$$\begin{aligned}\left|\Psi^{(2)}(t)\right\rangle_{\rm I} &= -\sum_{\gamma,\gamma_1\gamma'}\left\{\left[\frac{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+G_\gamma^{(2)}+G_\gamma^{(3)}\right)t}-\mathrm{e}^{-\mathrm{i}\left(E_{\gamma_1}+G_{\gamma_1}^{(2)}+G_{\gamma_1}^{(3)}\right)t}}{\left(E_\gamma-E_{\gamma_1}\right)^2}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma}\delta_{\gamma\gamma'}\right.\right.\\
&\quad+\left[\frac{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+G_\gamma^{(2)}+G_\gamma^{(3)}\right)t}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma'}\right)}-\frac{\mathrm{e}^{-\mathrm{i}\left(E_{\gamma_1}+G_{\gamma_1}^{(2)}+G_{\gamma_1}^{(3)}\right)t}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_{\gamma_1}-E_{\gamma'}\right)}\right.\\
&\quad\left.\left.+\frac{\mathrm{e}^{-\mathrm{i}\left(E_{\gamma'}+G_{\gamma'}^{(2)}+G_{\gamma'}^{(3)}\right)t}}{\left(E_\gamma-E_{\gamma'}\right)\left(E_{\gamma_1}-E_{\gamma'}\right)}\right]g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}\right\}a_{\gamma'}|\Phi^\gamma\rangle. \qquad (161)\end{aligned}$$

$$\begin{aligned}\left|\Psi^{(3)}(t)\right\rangle_{\rm I} &= \sum_{\gamma,\gamma_1,\gamma_2,\gamma'}\left\{\left[-\frac{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+G_\gamma^{(2)}\right)t}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma_2}\right)^2}-\frac{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+G_\gamma^{(2)}\right)t}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_\gamma-E_{\gamma_2}\right)}\right.\right.\\
&\quad\left.\left.+\frac{\mathrm{e}^{-\mathrm{i}\left(E_{\gamma_1}+G_{\gamma_1}^{(2)}\right)t}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_{\gamma_1}-E_{\gamma_2}\right)}-\frac{\mathrm{e}^{-\mathrm{i}\left(E_{\gamma_2}+G_{\gamma_2}^{(2)}\right)t}}{\left(E_\gamma-E_{\gamma_2}\right)^2\left(E_{\gamma_1}-E_{\gamma_2}\right)}\right]g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'}\right\}a_{\gamma'}|\Phi^\gamma\rangle\\
&\quad+\sum_{\gamma,\gamma'}\left\{-\sum_{\gamma_1}\left[\frac{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+G_\gamma^{(2)}\right)t}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma'}\right)^2}+\frac{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+G_\gamma^{(2)}\right)t}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_\gamma-E_{\gamma'}\right)}\right]g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma}g_1^{\gamma\gamma'}\right.\\
&\quad+\sum_{\gamma_1,\gamma_2}\left[\frac{\mathrm{e}^{-\mathrm{i}\left(E_\gamma+G_\gamma^{(2)}\right)t}\eta_{\gamma\gamma_2}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma_2}\right)\left(E_\gamma-E_{\gamma'}\right)}-\frac{\mathrm{e}^{-\mathrm{i}\left(E_{\gamma_1}+G_{\gamma_1}^{(2)}\right)t}\eta_{\gamma_1\gamma'}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_{\gamma_1}-E_{\gamma_2}\right)\left(E_{\gamma_1}-E_{\gamma'}\right)}\right.\\
&\quad\left.\left.+\frac{\mathrm{e}^{-\mathrm{i}\left(E_{\gamma_2}+G_{\gamma_2}^{(2)}\right)t}\eta_{\gamma\gamma_2}}{\left(E_\gamma-E_{\gamma_2}\right)\left(E_{\gamma_1}-E_{\gamma_2}\right)\left(E_{\gamma_2}-E_{\gamma'}\right)}\right]g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma'}\right\}a_{\gamma'}|\Phi^\gamma\rangle. \qquad (162)\end{aligned}$$

D. Summary

Obviously, our improved form of perturbed solution of dynamics in the third order approximation is

$$|\Psi(t)\rangle = \sum_{i=0}^{3}\left|\Psi^{(i)}(t)\right\rangle_{\rm I}+\mathcal{O}(H_1^4). \qquad (163)$$

However, this solution including the contributions from the whole $A_l^{\gamma\gamma'}(t\mathrm{e})$, $A_l^{\gamma\gamma'}(t^2\mathrm{e})$ parts up to the fifth order approximation and the whole $A_l^{\gamma\gamma'}(t^2\mathrm{e})$, $A_l^{\gamma\gamma'}(t^3\mathrm{e})$ parts in the sixth order approximation. After considering the contractions and anti-contractions, we see the result corresponds to the replacement

$$\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}\to\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma_i}t}, \qquad (164)$$

in the $A_l^{\gamma\gamma'}(\mathrm{e})$ part, where

$$\widetilde{E}_{\gamma_i}=E_{\gamma_i}+\sum_{a=2}G_{\gamma_i}^{(a)}, \qquad (165)$$

$i=0,1,2,\cdots$, and $\gamma_0=\gamma$. Although the upper bound of summation index a is different from the approximation order in the finished calculations, we can conjecture it may be taken to at least 5 based on the consideration from the physical concept and mathematical symmetry. For $a\geq 5$, their forms should be similar. From our point of view, such form is so delicate that its form happens impossibly by accident. Perhaps, there is a fundamental formula within it. Nevertheless, we have no idea of how to prove it strictly and generally at present.

Actually, as soon as we carry out further calculations, we can include the contributions from higher order approximations. Moreover, these calculations are not difficult and are programmable since we only need to calculate the limitation and summation. Therefore, the advantages of our solution have been made clear in our improved forms of perturbed solution of dynamics above. In other words, they offer clear evidences to show it is better than the existing method in the precision and efficiency. In the following several sections, we will clearly demonstrate these problems.

VIII. IMPROVED TRANSITION PROBABILITY AND REVISED FERMI'S GOLDEN RULE

One of the interesting applications of our solution is the calculation of transition probability in a general time-independent quantum system. It improves the well-known conclusion because our solution include the contributions from the high order approximation. Moreover, in terms of our improved forms of perturbed solution, it is easy to obtain the high order transition probability. In addition, for the case of sudden perturbation, our scheme is also suitable. Since considering time-independent system, it is not necessary to using the interaction picture.

Let us start with the following perturbed expansion of state evolution with time t,

$$|\Psi(t)\rangle = \sum_{\gamma} c_{\gamma}(t)|\Phi^{\gamma}\rangle = \sum_{l=0}^{\infty}\sum_{\gamma} c_{\gamma}^{(n)}(t)|\Phi^{\gamma}\rangle. \tag{166}$$

When we take the initial state as $\left|\Phi^{\beta}\right\rangle$, from our improved first order perturbed solution, we immediately obtain

$$c_{\gamma,\mathrm{I}}^{(1)} = \frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma}t} - \mathrm{e}^{-\mathrm{i}\widetilde{E}_{\beta}t}}{E_{\gamma} - E_{\beta}} g_1^{\gamma\beta}, \tag{167}$$

where

$$\widetilde{E}_{\gamma_i} = E_{\gamma_i} + G_{\gamma_i}^{(2)} + G_{\gamma_i}^{(3)} + G_{\gamma_i}^{(4)}. \tag{168}$$

Here, we use the subscript "I" for distinguishing it and the usual result. Omitting a unimportant phase factor $\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma}t}$, we can rewrite it as

$$c_{\gamma,\mathrm{I}}^{(1)} = \frac{g_1^{\gamma\beta}}{E_{\gamma} - E_{\beta}}\left(1 - \mathrm{e}^{\mathrm{i}\widetilde{\omega}_{\gamma\beta}t}\right), \tag{169}$$

where

$$\widetilde{\omega}_{\gamma\beta} = \widetilde{E}_{\gamma} - \widetilde{E}_{\beta}. \tag{170}$$

Obviously it is different from the well known conclusion

$$c_{\gamma}^{(1)} = \frac{g_1^{\gamma\beta}}{E_{\gamma} - E_{\beta}}\left(1 - \mathrm{e}^{\mathrm{i}\omega_{\gamma\beta}t}\right), \tag{171}$$

where

$$\omega_{\gamma\beta} = E_{\gamma} - E_{\beta}. \tag{172}$$

Therefore, our result contains the partial contributions from the high order approximation.

Considering the transition probability from $\left|\Phi^{\beta}\right\rangle$ to $|\Phi^{\gamma}\rangle$ after time T, we have

$$P_{\mathrm{I}}^{\gamma\beta}(t) = \frac{\left|g_1^{\gamma\beta}\right|^2}{\omega_{\gamma\beta}^2}\left|1 - \mathrm{e}^{\mathrm{i}\widetilde{\omega}_{\gamma\beta}T}\right|^2 = \left|g_1^{\gamma\beta}\right|^2 \frac{\sin^2\left(\widetilde{\omega}_{\gamma\beta}T/2\right)}{\left(\omega_{\gamma\beta}/2\right)^2}. \tag{173}$$

In terms of the relation

$$\sin^2 x - \sin^2 y = \frac{1}{2}\left[\cos(2y) - \cos(2x)\right], \tag{174}$$

we have the revision part of transition probability

$$\triangle P_{\mathrm{I}}^{\gamma\beta}(t) = 2\left|g_1^{\gamma\beta}\right|^2 \frac{\cos\left(\omega_{\gamma\beta}T\right) - \cos\left(\widetilde{\omega}_{\gamma\beta}T\right)}{\left(\omega_{\gamma\beta}\right)^2}. \tag{175}$$

If plotting

$$\frac{\sin^2\left(\widetilde{\omega}_{\gamma\beta}T/2\right)}{\left(\omega_{\gamma\beta}/2\right)^2} = \left(\frac{\widetilde{\omega}_{\gamma\beta}}{\omega_{\gamma\beta}}\right)^2 \frac{\sin^2\left(\widetilde{\omega}_{\gamma\beta}T/2\right)}{\left(\widetilde{\omega}_{\gamma\beta}/2\right)^2}, \tag{176}$$

we can see that it has a well-defined peak centered at $\widetilde{\omega}_{\gamma\beta}=0$. Just as what has been done in the usual case, we can extend the integral range as $-\infty\rightarrow\infty$. Thus, the revised Fermi's golden rule

$$w=w_{\mathrm{F}}+\triangle w, \tag{177}$$

where the usual Fermi's golden rule is [6]

$$w_{\mathrm{F}}=2\pi\rho(E_\beta)\left|g_1^{\gamma\beta}\right|^2, \tag{178}$$

in which w means the transition velocity, $\rho(E_\gamma)$ is the density of final state and we have used the integral formula

$$\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}=\pi. \tag{179}$$

while the revision part is

$$\triangle w=2\int_{-\infty}^{\infty}\mathrm{d}E_\gamma\rho\left(E_\gamma\right)\left|g_1^{\gamma\beta}\right|^2\frac{\cos\left(\omega_{\gamma\beta}T\right)-\cos\left(\widetilde{\omega}_{\gamma\beta}T\right)}{T\left(\omega_{\gamma\beta}\right)^2}. \tag{180}$$

It is clear that $\widetilde{\omega}_{\gamma\beta}$ is a function of E_γ, and then a function of $\omega_{\gamma\beta}$. For simplicity, we only consider $\widetilde{\omega}_{\gamma\beta}$ to its second order approximation, that is

$$\widetilde{\omega}_{\gamma\beta}=\widetilde{\omega}\left(\omega_{\gamma\beta}\right)=\omega_{\gamma\beta}+\sum_{\gamma_1}\left[\frac{|g_1^{\gamma\gamma_1}|^2}{\omega_{\gamma\beta}-\omega_{\gamma_1\beta}}-\frac{\left|g_1^{\beta\gamma_1}\right|^2}{\omega_{\beta\gamma_1}}\right]+\mathcal{O}(H_1^3). \tag{181}$$

Again based on $\mathrm{d}E_\gamma=\mathrm{d}\omega_{\gamma\beta}$, we have

$$\triangle w=2\int_{-\infty}^{\infty}\mathrm{d}\omega_{\gamma\beta}\rho\left(\omega_{\gamma\beta}+E_\beta\right)\left|g_1^{\gamma\beta}\right|^2\frac{\cos\left[\omega_{\gamma\beta}T\right]-\cos\left[\widetilde{\omega}\left(\omega_{\gamma\beta}\right)T\right]}{T\left(\omega_{\gamma\beta}\right)^2}. \tag{182}$$

It seems it is not easy to deduce the general form of this integral. In order to simplify it, we can use the fact that $\widetilde{\omega}_{\gamma\beta}-\omega_{\gamma\beta}$ is a smaller quantity since

$$\triangle\omega_{\gamma\beta}=\widetilde{\omega}_{\gamma\beta}-\omega_{\gamma\beta}=\sum_{i=2}^{4}\left(G_\gamma^{(i)}-G_\beta^{(i)}\right). \tag{183}$$

For example, we can approximatively take

$$\cos\left(\omega_{\gamma\beta}T\right)-\cos\left(\widetilde{\omega}_{\gamma\beta}T\right)\approx T\left(\widetilde{\omega}_{\gamma\beta}-\omega_{\gamma\beta}\right)\sin\left(\widetilde{\omega}_{\gamma\beta}T-\omega_{\gamma\beta}T\right), \tag{184}$$

then calculate the integral. We will study it in our other manuscript (in preparing).

Obviously, the revision comes from the contributions of high order approximations. The physical effect resulted from our solution, whether is important or unimportant, should be investigated in some concrete quantum systems. We will discuss them in our future manuscripts (in preparing).

It is clear that the relevant results can be obtained from the usual results via replacing $\omega_{\gamma\beta}$ in the exponential power by using $\widetilde{\omega}_{\gamma\beta}$. To save the space, we do not intend to mention them here.

In fact, there is not any difficult to obtain the second- and three order transition probability in terms of our improved forms of perturbed solution in the previous section. More higher order transition probability can be given effectively and accurately by our scheme.

IX. IMPROVED FORMS OF PERTURBED ENERGY AND PERTURBED STATE

Now we study how to calculate the improved forms of perturbed energy and perturbed state. For simplicity, we only study them concerning the improved second order approximation. Based on the experience from the skill one in Sec. VI, we can, in fact, set a new $\widetilde{E}$ and then use the technology in the usual perturbative theory. That is, we denote

$$\widetilde{E}_{\gamma_i}=E_{\gamma_i}+G_{\gamma_i}^{(2)}+G_{\gamma_i}^{(3)}. \tag{185}$$

$$
\begin{aligned}
|\Psi(t)\rangle \;=\; \sum_{\gamma,\gamma'}\Bigg\{ & \mathrm{e}^{-\mathrm{i}\widetilde{E}_\gamma t}\delta_{\gamma\gamma'} + \left[\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_\gamma t}-\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma'} t}}{E_\gamma - E_{\gamma'}}\right] g_1^{\gamma\gamma'} - \sum_{\gamma_1}\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_\gamma t}-\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma_1} t}}{\left(E_\gamma - E_{\gamma_1}\right)^2} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma}\delta_{\gamma\gamma'} \\
& + \sum_{\gamma_1}\left[\frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_\gamma t}}{\left(E_\gamma - E_{\gamma_1}\right)\left(E_\gamma - E_{\gamma'}\right)} - \frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma_1} t}}{\left(E_\gamma - E_{\gamma_1}\right)\left(E_{\gamma_1} - E_{\gamma'}\right)}\right. \\
& \left. + \frac{\mathrm{e}^{-\mathrm{i}\widetilde{E}_{\gamma'} t}}{\left(E_\gamma - E_{\gamma'}\right)\left(E_{\gamma_1} - E_{\gamma'}\right)}\right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma}\eta_{\gamma\gamma'}\Bigg\} a_{\gamma'}|\Phi^\gamma\rangle + \mathcal{O}(H_1^3).
\end{aligned} \tag{186}
$$

Because that

$$
|\Psi(t)\rangle_{\mathrm{I}} = \sum_{\gamma,\gamma'} \mathrm{e}^{-\mathrm{i}E_T t}\delta_{\gamma\gamma'} a_{\gamma'}|\Phi^\gamma\rangle, \tag{187}
$$

we have

$$
\begin{aligned}
E_T a_\gamma \;=\; & \widetilde{E}_\gamma a_\gamma + \sum_{\gamma'}\Bigg\{\frac{\widetilde{E}_\gamma - \widetilde{E}_{\gamma'}}{E_\gamma - E_{\gamma'}} g_1^{\gamma\gamma'} - \sum_{\gamma_1}\frac{\widetilde{E}_\gamma - \widetilde{E}_{\gamma_1}}{\left(E_\gamma - E_{\gamma_1}\right)^2} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma}\delta_{\gamma\gamma'} \\
& + \sum_{\gamma_1}\left[\frac{\widetilde{E}_\gamma}{\left(E_\gamma - E_{\gamma_1}\right)\left(E_\gamma - E_{\gamma'}\right)} - \frac{\widetilde{E}_{\gamma_1}}{\left(E_\gamma - E_{\gamma_1}\right)\left(E_{\gamma_1} - E_{\gamma'}\right)}\right. \\
& \left. + \frac{\widetilde{E}_{\gamma'}}{\left(E_\gamma - E_{\gamma'}\right)\left(E_{\gamma_1} - E_{\gamma'}\right)}\right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma}\eta_{\gamma\gamma'}\Bigg\} a_{\gamma'}.
\end{aligned} \tag{188}
$$

In the usual perturbation theory, H_1 is taken as a perturbed part with the form

$$
H_1 = \lambda \widetilde{H}_1, \tag{189}
$$

where λ is a real number that is called the perturbation parameter. It must be emphasized that $\widetilde{E}_{\gamma_i}$ can be taken as explicitly independent perturbed parameter λ, because we introduce λ as a formal multiplier after redefinition. This way has been seen in our skill one. Without loss of generality, we further take H_1 only with the nondiagonal form, that is

$$
H_1^{\gamma_1\gamma_2} = g_1^{\gamma_1\gamma_2} = \lambda \widetilde{g}_1^{\gamma_1\gamma_2}. \tag{190}
$$

Then, we expand both the desired expansion coefficients a_γ and the energy eigenvalues E_T in a power series of perturbation parameter λ:

$$
E_T = \sum_{l=0}^{\infty} \lambda^l E_{T,\mathrm{I}}^{(l)}, \tag{191}
$$

$$
a_\gamma = \sum_{l=0}^{\infty} \lambda^l a_{\gamma;\mathrm{I}}^{(l)}. \tag{192}
$$

1. *Improved 0th approximation*

If we set $\lambda = 0$, eq.(188) yields

$$
E_{T,\mathrm{I}}^{(0)} a_{\gamma;\mathrm{I}}^{(0)} = \widetilde{E}_\gamma a_{\gamma;\mathrm{I}}^{(0)}, \tag{193}
$$

where γ runs over all levels. Actually, let us focus on the level $\gamma = \beta$, then

$$
E_{T,\mathrm{I}}^{(0)} = \widetilde{E}_\beta. \tag{194}
$$

When the initial state is taken as $\left|\Phi^\beta\right\rangle$,

$$
a_{\gamma;\mathrm{I}}^{(0)} = \delta_{\gamma\beta}. \tag{195}
$$

Obviously, the improved form of perturbed energy is different from the results in the usual perturbative theory because it includes the contributions from the higher order approximation. However, the so-call improved form of perturbed state is the same as the usual result.

2. Improved 1st approximation

Again from eq.(188) it follows that

$$E_{T,\mathrm{I}}^{(0)}a_{\gamma;\mathrm{I}}^{(1)}+E_{T,\mathrm{I}}^{(1)}a_{\gamma;\mathrm{I}}^{(0)}=\widetilde{E}_{\gamma}a_{\gamma;\mathrm{I}}^{(1)}+\sum_{\gamma'}\frac{\widetilde{E}_{\gamma}-\widetilde{E}_{\gamma'}}{E_{\gamma}-E_{\gamma'}}\widetilde{g}_1^{\gamma\gamma'}a_{\gamma';\mathrm{I}}^{(0)}. \tag{196}$$

When $\gamma=\beta$, it is easy to obtain

$$E_{T,\mathrm{I}}^{(1)}=0. \tag{197}$$

If $\gamma\neq\beta$, then

$$a_{\gamma;\mathrm{I}}^{(1)}=-\frac{1}{\left(E_{\gamma}-E_{\gamma^{\beta}}\right)}\widetilde{g}_1^{\gamma\beta}. \tag{198}$$

It is clear that the first order results are the same as the one in the usual perturbative theory.

3. Improved 2nd approximation

Likewise, the following equation

$$\begin{aligned}E_{T,\mathrm{I}}^{(2)}a_{\gamma;\mathrm{I}}^{(0)}+E_{T,\mathrm{I}}^{(1)}a_{\gamma;I}^{(1)}+E_{T,\mathrm{I}}^{(0)}a_{\gamma;\mathrm{I}}^{(2)}&=\widetilde{E}_{\gamma}a_{\gamma;\mathrm{I}}^{(2)}+\sum_{\gamma'}\frac{\widetilde{E}_{\gamma}-\widetilde{E}_{\gamma'}}{E_{\gamma}-E_{\gamma'}}\widetilde{g}_1^{\gamma\gamma'}a_{\gamma';\mathrm{I}}^{(1)}\\&-\sum_{\gamma_1,\gamma'}\frac{\widetilde{E}_{\gamma}-\widetilde{E}_{\gamma_1}}{\left(E_{\gamma}-E_{\gamma_1}\right)^2}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma}\delta_{\gamma\gamma'}a_{\gamma';\mathrm{I}}^{(0)}+\sum_{\gamma_1,\gamma'}\left[\frac{\widetilde{E}_{\gamma}}{\left(E_{\gamma}-E_{\gamma_1}\right)\left(E_{\gamma}-E_{\gamma'}\right)}\right.\\&\left.-\frac{\widetilde{E}_{\gamma_1}}{\left(E_{\gamma}-E_{\gamma_1}\right)\left(E_{\gamma_1}-E_{\gamma'}\right)}+\frac{\widetilde{E}_{\gamma'}}{\left(E_{\gamma}-E_{\gamma'}\right)\left(E_{\gamma_1}-E_{\gamma'}\right)}\right]g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma}\eta_{\gamma\gamma'}a_{\gamma';\mathrm{I}}^{(0)}.\end{aligned} \tag{199}$$

is obtained and it yields

$$E_{T;\mathrm{I}}^{(2)}=0, \tag{200}$$

if we take $\gamma=\beta$. When $\gamma\neq\beta$, we have

$$a_{\gamma;\mathrm{I}}^{(2)}=\sum_{\gamma_1}\frac{1}{\left(E_{\gamma}-E_{\beta}\right)\left(E_{\gamma_1}-E_{\beta}\right)}\widetilde{g}_1^{\gamma\gamma_1}\widetilde{g}_1^{\gamma_1\beta}\eta_{\gamma\beta}. \tag{201}$$

It is consistent with the nondiagonal part of usual result. In fact, since we have taken $H_1^{\gamma\gamma'}$ to be nondiagonal, it does not have a diagonal part. However, we think its form is more appropriate. In addition, we do not consider the revision part introduced by normalization. While $E_{T;\mathrm{I}}^{(2)}=0$ is a new result.

4. Summary

Now we can see, up to the improved second order approximation:

$$E_{T,\beta}\approx\widetilde{E}_{\beta}=E_{\beta}+G_{\beta}^{(2)}+G_{\beta}^{(3)}. \tag{202}$$

Comparing with the usual, they are consistent at the former two orders. It is not strange since the physical law is the same. However, our improved form of perturbed energy contains a third order term. In other words, our solution might be effective in order to obtain the contribution from high order approximation. The possible physical reason is that a redefined form of the solution is obtained.

In special, when we allow the $H_1^{\gamma\gamma'}$ to have the diagonal elements, the improved second order approximation becomes

$$E_{T,\beta} \approx E_\beta + h_1^\beta + G_\beta^{(2)} + G_\beta^{(3)}. \tag{203}$$

Likewise, if we redefine

$$\widetilde{E}_{\gamma_i} = E_{\gamma_i} + G_{\gamma_i}^{(2)} + G_{\gamma_i}^{(3)} + G_{\gamma_i}^{(4)}. \tag{204}$$

Thus, only consider the first order approximation, we can obtain

$$E_{T,\beta} \approx E_\beta + h_1^\beta + G_\beta^{(2)} + G_\beta^{(3)} + G_\beta^{(4)}. \tag{205}$$

In fact, the reason is our conjecture in the previous section. The correct form of redefined $\widetilde{E}_{\gamma_i}$ should be

$$E_{T,\beta} \approx E_\beta + h_1^\beta + G_\beta^{(2)} + G_\beta^{(3)} + G_\beta^{(4)} + G_\beta^{(5)} + \cdots. \tag{206}$$

This implies that our improved scheme includes the partial even whole significant contributions from the high order approximations. In addition, based on the fact that the improved second approximation is actually zero, it is possible that this implies our solution will fade down more rapidly than the solution in the usual perturbative theory.

Actually, the main advantage of our solution is in dynamical development. The contributions from the high order approximation play more important roles in the relevant physical problems such as the entanglement dynamics and decoherence process. For the improved perturbed energy, its high order part has obvious physical meaning. But, for the improved form of perturbed state, we find them to be the same.

X. EXAMPLE AND APPLICATION

In order to concretely illustrate that our exact solution and perturbative scheme are indeed more effective and more accurate, let us study an elementary example: two state system, which appears in the most of quantum mechanics textbooks. Its Hamiltonian can be written as

$$H = \begin{pmatrix} E_1 & V_{12} \\ V_{21} & E_2 \end{pmatrix}, \tag{207}$$

where we have used the the basis formed by the unperturbed energy eigenvectors, that is

$$\left|\Phi^1\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \left|\Phi^2\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{208}$$

In other words:

$$H_0|\Phi^\gamma\rangle = E_\gamma|\Phi^\gamma\rangle, \quad (\gamma = 1, 2) \tag{209}$$

where

$$H_0 = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}. \tag{210}$$

Thus, this means the perturbed Hamiltonian is taken as

$$H_1 = \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix}. \tag{211}$$

This two state system has the following eigen equation

$$H|\Psi^\gamma\rangle = E_\gamma^T|\Psi^\gamma\rangle. \tag{212}$$

It is easy to obtain its solution: corresponding eigenvectors and eigenvalues

$$\left|\Psi^1\right\rangle = \frac{1}{\sqrt{4|V|^2 + (\omega_{21} + \omega_{21}^T)^2}} \begin{pmatrix} \omega_{21} + \omega_{21}^T \\ -2V_{21} \end{pmatrix}, \tag{213}$$

$$\left|\Psi^2\right\rangle = \frac{1}{\sqrt{4|V|^2 + (\omega_{21} - \omega_{21}^T)^2}} \begin{pmatrix} \omega_{21} - \omega_{21}^T \\ -2V_{21} \end{pmatrix}; \tag{214}$$

$$
\begin{aligned}
E_1^T &= \frac{1}{2}\left(E_1+E_2-\omega_{21}^T\right), &(215)\\
E_2^T &= \frac{1}{2}\left(E_1+E_2+\omega_{21}^T\right); &(216)
\end{aligned}
$$

where $|V|=|V_{12}|=|V_{21}|$, $\omega_{21}=E_2-E_1$, $\omega_{21}^T=E_2^T-E_1^T=\sqrt{4|V|^2+\omega_{21}^2}$, and we have set $E_2>E_1$ without loss of generality.

Obviously the transition probability from state 1 to state 2 is

$$
\begin{aligned}
P^T(1\to 2) &= \left|\left\langle\Phi^2\middle|\mathrm{e}^{-\mathrm{i}Ht}\middle|\Phi^1\right\rangle\right|^2=\left|\sum_{\gamma_1,\gamma_2=1}^{2}\langle\Phi^2|\Psi^{\gamma_1}\rangle\langle\Psi^{\gamma_1}|\mathrm{e}^{-\mathrm{i}Ht}|\Psi^{\gamma_2}\rangle\langle\Psi^{\gamma_2}|\Phi^1\rangle\right|^2\\
&= |V|^2\frac{\sin^2\left(\omega_{21}^Tt/2\right)}{(\omega_{21}^T/2)^2}. &(217)
\end{aligned}
$$

In the usual perturbation theory, up to the second order approximations, the well-known the perturbed energies are

$$
\begin{aligned}
E_1^P &= E_1-\frac{|V|^2}{\omega_{21}}, &(218)\\
E_2^P &= E_1+\frac{|V|^2}{\omega_{21}}. &(219)
\end{aligned}
$$

While, under the first order approximation, the transition probability from state 1 to state 2 is

$$
P(1\to 2)=|V|^2\frac{\sin^2\left(\omega_{21}t/2\right)}{(\omega_{21}/2)^2}. \tag{220}
$$

Using our scheme, only to the first approximation, the corresponding the perturbed energies are

$$
\begin{aligned}
\widetilde{E}_1 &= E_1-\frac{|V|^2}{\omega_{21}}+\frac{|V|^4}{\omega_{21}^3}, &(221)\\
\widetilde{E}_2 &= E_1+\frac{|V|^2}{\omega_{21}}-\frac{|V|^4}{\omega_{21}^3}, &(222)
\end{aligned}
$$

where we have used the facts that

$$
\begin{aligned}
G_1^{(2)} &= -\frac{|V|^2}{\omega_{21}}=-G_2^{(2)}, &(223)\\
G_1^{(3)} &= G_2^{(3)}=0, &(224)\\
G_1^{(4)} &= \frac{|V|^4}{\omega_{21}^3}=-G_2^{(4)}. &(225)
\end{aligned}
$$

Obviously, under the first order approximation, our scheme yields the transition probability from state 1 to state 2 as

$$
P_{\rm I}(1\to 2)=|V|^2\frac{\sin^2\left(\widetilde{\omega}_{21}t/2\right)}{(\omega_{21}/2)^2}. \tag{226}
$$

where $\widetilde{\omega}_{21}=\widetilde{E}_2-\widetilde{E}_1$. Therefore we can say our scheme is more effective. Moreover, we notice that

$$
\begin{aligned}
E_1^T &= E_1-\frac{|V|^2}{\omega_{21}}+\frac{|V|^4}{\omega_{21}^3}+\mathcal{O}(|V|^6) &(227)\\
&= \widetilde{E}_1+\mathcal{O}(|V|^6) &(228)\\
&= E_1^P+\frac{|V|^4}{\omega_{21}^3}+\mathcal{O}(|V|^6), &(229)\\
\widetilde{E}_2 &= E_1+\frac{|V|^2}{\omega_{21}}-\frac{|V|^4}{\omega_{21}^3}+\mathcal{O}(|V|^6) &(230)\\
&= \widetilde{E}_2+\mathcal{O}(|V|^6) &(231)\\
&= E_2^P-\frac{|V|^4}{\omega_{21}^3}+\mathcal{O}(|V|^6). &(232)
\end{aligned}
$$

and

$$P^T(1\to 2) \;=\; |V|^2\frac{\sin^2(\omega_{21}t/2)}{(\omega_{21}/2)^2}+|V|^2\left[\frac{\sin(\omega_{21}t)}{2(\omega_{21}/2)^2}-\frac{\sin^2(\omega_{21}t/2)}{(\omega_{21}/2)^3}\right](\widetilde{\omega}_{21}-\omega_{21})+\mathcal{O}[(\widetilde{\omega}_{21}-\omega_{21})^2] \tag{233}$$

$$= P_{\rm I}(1\to 2)-|V|^2\frac{\sin^2(\omega_{21}t/2)}{(\omega_{21}/2)^3}(\widetilde{\omega}_{21}-\omega_{21})+\mathcal{O}[(\widetilde{\omega}_{21}-\omega_{21})^2] \tag{234}$$

$$= P(1\to 2)+|V|^2\left[\frac{\sin(\omega_{21}t)}{2(\omega_{21}/2)^2}-\frac{\sin^2(\omega_{21}t/2)}{(\omega_{21}/2)^3}\right](\widetilde{\omega}_{21}-\omega_{21})+\mathcal{O}[(\widetilde{\omega}_{21}-\omega_{21})^2]. \tag{235}$$

Therefore, we can say that our scheme is more accurate.

XI. CONCLUSION AND DISCUSSION

In the end, we would like to point out that our general and explicit solution (35) or its particular forms (42,43) of the Schrödinger equation in a general time-independent quantum system is clear and exact in form in spite of it is an infinity series. It must be emphasized that its distinguished feature is to first express the exact solution in a general time-independent quantum system by using the c-number matrix elements rather than an operator form. We can say that our exact solution is more explicit than the usual nonperturbed solution of the Schrödinger equation. Moreover, our deducing methods give up some preconditions used in the usual scheme. So we can say our solution is more general. Just as its explicit and general form, our exact solution not only has the mathematical delicateness, but also can contain more physical content, obtain the more efficiency and higher precision and result in new applications.

We obtain our exact solution based on our expansion formula of power of operator binomials and the matrix representation of time evolution operator, which is a new way to study the quantum dynamics and the perturbation theory. Its idea is different from the usual one. At our knowledge, this formula is first proposed and strictly proved. Besides its theoretical value in mathematics, we are sure it is interesting and important for expressing some useful operator formula in quantum physics.

In the process of deducing our exact solution, we prove an identity of fraction function. It should be interesting in mathematics. Perhaps, it has other applications to be expected finding.

By virtu of the idea of quantum field theory, we build the concepts of contraction and anti-contraction using in the limitation computation. Moreover we develop two skills to deal with an infinity series, in special, so-called dynamical rearrangement and summation. These methods have played important roles in our scheme. We believe they can be useful technologies in mathematical and physical calculation.

As an example, we give out the concrete form of solution when the solvable part of Hamiltonian is taken as the kinetic energy term in a general quantum system. Its aim is to account for the fact that our solution perhaps extends the applicable range of perturbation theory and lose its preconditions.

We can see that in our solution, every expanding coefficient (amplitude) before the basis vector $|\Phi^\gamma\rangle$ has some closed time evolution factors with exponential form, and includes all order approximations so that we can clearly understand the dynamical behavior of quantum systems. Although our solution is obtained in H_0 representation, its form in the other representation can be given by the representation transformation and the development factors with time t in the expanding coefficients (amplitude) do not change.

Because we obtain the general term form of time evolution of quantum state, it provides the probability considering the partial contributions from the high order even all of order approximations. After developing the contraction and anti-contraction skills and dynamical rearrangement and summation technology, we realize this probability by present a perturbative scheme. From our exact solution transferring to our perturbative scheme is physically reasonable and mathematically clear. This provides the guarantee achieving high efficiency and high precision. Through finding the improved forms of perturbed solutions of dynamics, we generally demonstrate this conclusion. Furthermore, we prove the correctness of this conclusion via calculating the improved form of transition probability, perturbed energy and perturbed state. Specially, we obtain the revised Fermi's golden rule. Moreover, we illustrate the advantages of our exact solution and perturbative scheme in an easy understanding example of two state system. All of this implies the physical reasons and evidences why our exact solution and perturbative scheme are formally explicit, actually calculable, operationally efficient, conclusively more accurate.

From the features of our solution, we believe that it will have interesting applications in the calculation of entanglement dynamics and decoherence process as well as the other physical quantities dependent on the expanding coefficients. Of course, our solution is an exact one, its advantages and features can not be fully revealed only via the perturbative method.

Undoubtedly, the formalization of physical theory often has its highly mathematical focus, but this can not cover its real motivations, potential applications and related conclusions in physics. Our exact solution and perturbative

scheme are just so. In order to account for what is more in our solution and reveal the relations and differences between our solution and the existing method, we compare our solution with the usual perturbation theory. We find their consistency and relations. In fact, our solution has finished the task to calculate the expanding coefficients of final state in H_0 representation and obtain the general term up to any order using our own method. But the usual perturbation theory only carries out this task from some given order approximation to the next order approximation step by step. In a sentence, more explicit and general feature of our exact solution can lead to more physical conclusions and applications.

It is worth pointing out that our solution, different from the time-dependent perturbation theory, presents the explicit solution of recurrence equation of expanding coefficients of final state in H_0 representation, but we pay for the price that H is not explicitly dependent on the time. In short, there is gain and there is lose.

In fact, a given lower order approximation of improved perturbation solution including the partial contributions from the higher order even all of order approximations is obtained by rearranging and summing a kind of terms, just like it has been done in the quantum field theory for the particular contributions over a given type of the Feynman's diagrams. It is emphasized that these contributions have to be significant in physics. Considering time development form is our physical idea and adding the high order approximations with the factors $t^a \mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}$, $(a = 1, 2, \cdots)$ to the improved lower order approximations definitely can advance the precision. Therefore, our dynamical rearrangement and summation technology is appropriate and reasonable from our view.

For a concrete example, except for some technological and calculational works, it needs the extensive physical background knowledge to account for the significance of related results. That is, since the difference of the related conclusions between our solution and the usual perturbation theory is in high order approximation parts, we have to study the revisions (difference) to find out whether they are important or unimportant to the studied problem. In addition, our conjecture about the perturbed energy is based on physical concept and mathematical consideration, it is still open at the strict sense. As to the degenerate case, except for the complicated expressions, there is no more new idea.

Based on the above statements, our results can be thought of as theoretical developments of quantum dynamics. The extension of our solution to mixed states is direct, and the extension of our solution to open systems see our manuscript [7].

It must be emphasized that the study on the time evolution operator plays a central role in quantum dynamics. From our point of view, one of the most main results in our method is to obtain a general term form of any order of H_1 (perturbative part of Hamiltonian) for the time evolution operation in the representation of H_0 (solvable part of Hamiltonian), however, the usual perturbation theory has not really finished it and only has an expression in the operator form. Because the universal significance of our new expression of time evolution operator, we wish that it will have more applications in quantum theory. Besides the above studies through the perturbative method, it is more interesting to apply our exact solution to the formalization study of quantum dynamics in order to further and more powerfully show the advantages of our exact solution.

In summary, we can say our present results are helpful to understand the theory of quantum dynamics and provide some powerful tools in the calculation of quantum dynamics. More investigations are on progressing.

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APPENDIX A: THE PROOF OF OUR IDENTITY

In this appendix, we would like to prove our identity (26). For simplicity in notation and universality, we replace the variables E_{γ_i} by x_i as well as $E[\gamma, l]$ by $x[l]$. It is clear that the common denominator $D(x[n])$ in the above expression (26) (the index l is replaced by n) reads

$$D(x[n]) = \prod_{i=1}^{n}\left[\prod_{j=i}^{n}(x_i - x_{j+1})\right], \tag{A1}$$

while the i-th numerator is

$$n_i(x[n], K) = \frac{D(x[n])}{d_i(x[n])} x_i^K, \tag{A2}$$

and the total numerator $N(x[n],K)$ is

$$N(x[n],K]=\sum_{i=1}^{n+1}n_i(x[n],K). \tag{A3}$$

In order to simplify our notation, we denote $n_i(x[n])=n_i(x[n],0)$. Again, introducing a new vector

$$\begin{aligned} x_1^D[n] &= \{x_2,x_3,\cdots,x_n,x_{n+1}\}, && \text{(A4)}\\ x_i^D[n] &= \{x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_{n+1}\}, && \text{(A5)}\\ x_{n+1}^D[n] &= \{x_1,x_2,\cdots,x_{n-1},x_n\}. && \text{(A6)}\end{aligned}$$

Obviously, $x_i^D(x[n])$ with n components is obtained by deleting the i-th component from $x[n]$. From the definition of $n_i(x[n])$, it follows that

$$n_i(x[n])=D(x_i^D[n]). \tag{A7}$$

Without loss of generality, for an arbitrary given i, we always can rewrite $x_i^D[n]=y[n-1]=\{y_1,y_2,\cdots,y_{n-1},y_n\}$ and consider the general expression of $D(y[n-1])$. It is easy to verify that

$$\begin{aligned} D(y[1])&=-y_2n_1(y[1])+y_1n_2(y[1]), && \text{(A8)}\\ D(y[2])&=y_2y_3n_1(y[2])-y_1y_3n_2(y[2])+y_1y_2n_3(y[2]). && \text{(A9)}\end{aligned}$$

Thus, by mathematical induction, we first assume that for $n\geq 2$,

$$D(y[n-1])=\sum_{i=1}^{n}(-1)^{(i-1)+(n-1)}p_i(y[n-1])n_i(y[n-1]), \tag{A10}$$

where we have defined

$$p_i(y[n-1])=\prod_{\substack{j=1\\ j\neq i}}^{n}y_j. \tag{A11}$$

As above, we have verified that the expression (A10) is valid for $n=2,3$. Then, we need to prove that the following expression

$$D(y[n])=\sum_{i=1}^{n+1}(-1)^{(i-1)+n}p_i(y[n])n_i(y[n]) \tag{A12}$$

is correct.

To our purpose, we start from the proof of a conclusion of the precondition (A10) as the following:

$$\sum_{i=1}^{n+1}(-1)^{i-1}n_i(y[n])=0. \tag{A13}$$

According to the relation (A7) and substituting the precondition (A10), we have

$$\begin{aligned} \sum_{i=1}^{n+1}(-1)^{i-1}n_i(y[n]) &= \sum_{i=1}^{n+1}(-1)^{i-1}D(y_i^D[n])\\ &= \sum_{i=1}^{n+1}(-1)^{i-1}\left[\sum_{j=1}^{n}(-1)^{(j-1)+(n-1)}p_j(y_i^D[n])n_j(y_i^D[n])\right]\\ &= (-1)^{n-1}\sum_{j=1}^{n}\sum_{i=1}^{n+1}(-1)^{i+j}p_j(y_i^D[n])n_j(y_i^D[n]). \end{aligned} \tag{A14}$$

Based on the definitions of $p_i(x[n])$ and $n_i(x[n])$, we find that

$$p_j(y_i^D[n]) = \begin{cases} p_{i-1}(y_j^D[n]) & (\text{If } i > j) \\ p_i(y_{j+1}^D[n]) & (\text{If } i \leq j) \end{cases}, \tag{A15}$$

$$n_j(y_i^D[n]) = \begin{cases} n_{i-1}(y_j^D[n]) & (\text{If } i > j) \\ n_i(y_{j+1}^D[n]) & (\text{If } i \leq j) \end{cases}. \tag{A16}$$

Thus, the right side of eq.(A14) becomes

$$\begin{aligned} & (-1)^{n-1}\sum_{j=1}^{n}\sum_{i=1}^{n+1}(-1)^{i+j}p_j(y_i^D[n])n_j(y_i^D[n]) \\ = & \ (-1)^{n-1}\sum_{j=1}^{n}\sum_{i=j+1}^{n+1}(-1)^{i+j}p_j(y_i^D[n])n_j(y_i^D[n]) \\ & +(-1)^{n-1}\sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{i+j}p_j(y_i^D[n])n_j(y_i^D[n]) \\ = & \ (-1)^{n-1}\sum_{j=1}^{n}\sum_{i=j+1}^{n+1}(-1)^{i+j}p_{i-1}(y_j^D[n])n_{i-1}(y_j^D[n]) \\ & +(-1)^{n-1}\sum_{j=1}^{n}\sum_{i=1}^{j}(-1)^{i+j}p_i(y_{j+1}^D[n])n_i(y_{j+1}^D[n]). \end{aligned} \tag{A17}$$

Setting that $i-1 \to i$ for the first term and $j+1 \to j$ for the second term in the right side of the above equation, we obtain

$$\begin{aligned} & (-1)^{n-1}\sum_{j=1}^{n}\sum_{i=1}^{n+1}(-1)^{i+j}p_j(y_i^D[n])n_j(y_i^D[n]) \\ = & \ (-1)^{n-1}\sum_{j=1}^{n}\sum_{i=j}^{n}(-1)^{i+j-1}p_i(y_j^D[n])n_i(y_j^D[n]) \\ & +(-1)^{n-1}\sum_{j=2}^{n+1}\sum_{i=1}^{j-1}(-1)^{i+j-1}p_i(y_j^D[n])n_i(y_j^D[n]) \\ = & \ (-1)^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}(-1)^{i+j}p_i(y_j^D[n])n_i(y_j^D[n]) \\ & +(-1)^{n}\sum_{i=1}^{n+1}(-1)^{i+(n+1)}p_i(y_{n+1}^D[n])n_i(y_{n+1}^D[n]). \end{aligned} \tag{A18}$$

Again, setting that $i \leftrightarrow j$ in the right side of the above equation gives

$$\begin{aligned} & (-1)^{n-1}\sum_{j=1}^{n}\sum_{i=1}^{n+1}(-1)^{i+j}p_j(y_i^D[n])n_j(y_i^D[n]) \\ & = (-1)^{n}\sum_{j=1}^{n}\sum_{i=1}^{n+1}(-1)^{i+j}p_j(y_i^D[n])n_j(y_i^D[n]). \end{aligned} \tag{A19}$$

Thus, it implies that

$$(-1)^{n-1}\sum_{j=1}^{n}\sum_{i=1}^{n+1}(-1)^{i+j}p_j(y_i^D[n])n_j(y_i^D[n]) = 0. \tag{A20}$$

From the relation (A14) it follows that the identity (A13) is valid.

Now, let us back to the proof of the expression (A12). Since the definition of $D(y[n])$, we can rewrite it as

$$D(y[n]) = D(y[n-1])d_{n+1}(y[n]). \tag{A21}$$

Substituting our precondition (A10) yields

$$D(y[n]) \;=\; \sum_{i=1}^{n}(-1)^{(i-1)+(n-1)}p_i(y[n-1])n_i(y[n-1])d_{n+1}(y[n]). \tag{A22}$$

Note that

$$\begin{aligned} n_i(y[n-1])d_{n+1}(y[n]) &= n_i(y[n])(y_i - y_{n+1}), & \text{(A23)}\\ p_i(y[n-1])y_{n+1} &= p_i(y[n]), & \text{(A24)}\\ p_i(y[n-1])y_i &= p_{n+1}(y[n]), & \text{(A25)} \end{aligned}$$

we have that

$$\begin{aligned} D(y[n]) &= \sum_{i=1}^{n}(-1)^{(i-1)+n}p_i(y[n])n_i(y[n]) + \sum_{i=1}^{n}(-1)^{(i-1)+(n-1)}p_{n+1}(y[n])n_i(y[n]) \\ &= \sum_{i=1}^{n}(-1)^{(i-1)+n}p_i(y[n])n_i(y[n]) + (-1)^{n-1}p_{n+1}(y[n])\left[\sum_{i=1}^{n}(-1)^{i-1}n_i(y[n])\right] \\ &= \sum_{i=1}^{n}(-1)^{(i-1)+n}p_i(y[n])n_i(y[n]) + (-1)^{n-1}p_{n+1}(y[n])\left[-(-1)^{n}n_{n+1}(y[n])\right]. \end{aligned} \tag{A26}$$

In the last equality we have used the conclusion (A13) of our precondition, that is

$$\sum_{i=1}^{n}(-1)^{i-1}n_i(y[n]) = -(-1)^{n}n_{n+1}(y[n]). \tag{A27}$$

Thus, eq.(A26) becomes

$$D(y[n]) = \sum_{i=1}^{n+1}(-1)^{(i-1)+n}p_i(y[n])n_i(y[n]). \tag{A28}$$

The needed expression (A12) is proved by mathematical induction. That is, we have proved that for any $n \geq 1$, the expression (A12) is valid.

Since our proof of the conclusion (A13) of our precondition is independent of n ($n \geq 1$), we can, in the same way, prove that the identity (A13) is correct for any $n \geq 1$.

Now, let us prove our identity (26). Obviously, when $K = 0$ we have

$$\sum_{i=1}^{n+1}(-1)^{i-1}\frac{1}{d_i(x[n])} = \frac{1}{D(x[n])}\left[\sum_{i=1}^{n+1}(-1)^{i-1}n_i(x[n])\right] = 0, \tag{A29}$$

where we have used the fact the identity (A13) is valid for any $n \geq 1$. Furthermore, we extend the definition domain of $C_n^K(x[n])$ from $K \geq n$ to $K \geq 0$, and still write its form as

$$C_n^K(x[n]) = \sum_{i=1}^{n+1}(-1)^{i-1}\frac{x_i^K}{d_i(x[n])}. \tag{A30}$$

Obviously, eq.(A29) means

$$C_n^0(x[n]) = 0, \quad (n \geq 1). \tag{A31}$$

In order to consider the cases when $K \neq 0$, by using of $d_i(x[n])(x_i - x_{n+2}) = d_i(x[n+1])$ $(i \leq n+1)$, we obtain

$$\begin{aligned} C_n^K(x[n]) &= \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i^K}{d_i(x[n])} \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i^K}{d_i(x[n])} \left(\frac{x_i - x_{n+2}}{x_i - x_{n+2}} \right) \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i^{K+1}}{d_i(x[n+1])} \\ &\quad - x_{n+2} \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i^K}{d_i(x[n+1])} \\ &= C_{n+1}^{K+1}(x[n+1]) - (-1)^{(n+2)-1} \frac{x_{n+2}^{K+1}}{d_{n+2}(x[n+1])} \\ &\quad - x_{n+2} \sum_{i=1}^{n+1} (-1)^{i-1} \frac{x_i^K}{d_i(x[n+1])}. \end{aligned} \tag{A32}$$

It follows the recurrence equation as the following

$$C_n^K(x[n]) = C_{n+1}^{K+1}(x[n+1]) - x_{n+2} C_{n+1}^K(x[n+1]). \tag{A33}$$

It implies that since $C_n^0(x[n]) = 0$ for any $n \geq 1$, then $C_{n+1}^1(x[n+1]) = 0$ for any $n \geq 1$ or $C_n^1(x[n]) = 0$ for any $n \geq 2$; since $C_n^1(x[n]) = 0$ for any $n \geq 2$, then $C_{n+1}^2(x[n+1]) = 0$ for any $n \geq 2$ or $C_n^2(x[n]) = 0$ for any $n \geq 3$; $\cdots$, since $C_n^k(x[n]) = 0$ for any $n \geq (k+1)$ $(k \geq 0)$, then $C_{n+1}^{k+1}(x[n+1]) = 0$ for any $n \geq (k+1)$ or $C_n^{k+1}(x[n]) = 0$ for any $n \geq (k+2)$; $\cdots$. In fact, the mathematical induction tells us this result. Obviously

$$C_n^K(x[n]) = 0, \quad (\text{If } 0 \leq K < n). \tag{A34}$$

Taking $K = n$ in eq.(A33) and using eq.(A34), we have

$$C_n^n(x[n]) = C_{n+1}^{n+1}(x[n+1]) = 1, \quad (n \geq 1), \tag{A35}$$

where we have used the fact that $C_1^1(x[1]) = 1$. Therefore, the proof of our identity (26) is finally finished.

APPENDIX B: THE CALCULATIONS OF THE HIGH ORDER TERMS

Since we have taken the H_1 only with the non diagonal part, it is enough to calculate the contributions from them. In Sec. VI the contributions from the first, second and third order approximations. In this appendix, we would like to find the contributions from the fourth to the sixth order approximations. The used calculation technologies are mainly to find the limitation, dummy index changing and summation, as well as the replacement $g_1^{\gamma_i\gamma_j}\eta_{\gamma_i\gamma_j} = g_1^{\gamma_i\gamma_j}$ since $g_1^{\gamma_i\gamma_j}$ has been nondiagonal. They are not difficult, but are a little lengthy.

1. $l = 4$ case

For the fourth order approximation, its contributions from the first decompositions consists of eight terms:

$$\begin{aligned} A_4^{\gamma\gamma'} &= A_4^{\gamma\gamma'}(ccc) + A_4^{\gamma\gamma'}(ccn) + A_4^{\gamma\gamma'}(cnc) + A_4^{\gamma\gamma'}(ncc) \\ &\quad + A_4^{\gamma\gamma'}(cnn) + A_4^{\gamma\gamma'}(ncn) + A_4^{\gamma\gamma'}(nnc) + A_4^{\gamma\gamma'}(nnn). \end{aligned} \tag{B1}$$

We will see that the former four terms have no the nontrivial second contractions, the fifth and seven terms have one nontrivial second contraction,

$$A_4^{\gamma\gamma'}(cnn) = A_4^{\gamma\gamma'}(cnn, kc) + A_4^{\gamma\gamma'}(cnn; kn), \tag{B2}$$

$$A_4^{\gamma\gamma'}(ncn) = A_4^{\gamma\gamma'}(ncn, c) + A_4^{\gamma\gamma'}(ncn, n), \tag{B3}$$

$$A_4^{\gamma\gamma'}(nnc) = A_4^{\gamma\gamma'}(nnc, ck) + A_4^{\gamma\gamma'}(nnc, nk). \tag{B4}$$

In addition, the last term in eq.(B1) has two nontrivial second contractions, thus

$$A_4^{\gamma\gamma'}(nnn) = A_4^{\gamma\gamma'}(nnn,cc) + A_4^{\gamma\gamma'}(nnn,cn) A_4^{\gamma\gamma'}(nnn,nc) + A_4^{\gamma\gamma'}(nnn,nn), \tag{B5}$$

its fourth term has also the third contraction

$$A_4^{\gamma\gamma'}(nnn,nn) = A_4^{\gamma\gamma'}(nnn,nn,c) + A_4^{\gamma\gamma'}(nnn,nn,n). \tag{B6}$$

All together, we have the fifteen expressions of the contributions from whole contractions in the fourth order approximation.

First, let us calculate the former four terms only with the first contractions and anti-contractions, that is, with more than two δ functions (or less than two η functions)

$$\begin{aligned}
A_4^{\gamma\gamma'}(ccc) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4} g_1^{\gamma_j\gamma_{j+1}}\right]\left(\prod_{k=1}^{3}\delta_{\gamma_k\gamma_{k+2}}\right)\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1}\left[\frac{3\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^4} - \frac{3\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma-E_{\gamma_1})^4} - (-\mathrm{i}t)\frac{2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^3}\right. \\
&\quad \left. -(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma-E_{\gamma_1})^3} + \frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2}\right]\left|g_1^{\gamma\gamma_1}\right|^4\delta_{\gamma\gamma'}.
\end{aligned} \tag{B7}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(ccn) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4} g_1^{\gamma_j\gamma_{j+1}}\right]\delta_{\gamma_1\gamma_3}\delta_{\gamma_2\gamma_4}\eta_{\gamma_3\gamma_5}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1}\left[-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma'})^2} - \frac{2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^3(E_\gamma-E_{\gamma'})}\right. \\
&\quad -\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma'})^2} + \frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma-E_{\gamma_1})^3(E_{\gamma_1}-E_{\gamma'})} \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma'})} \\
&\quad \left. +(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma'})}\right]\left|g_1^{\gamma\gamma_1}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}.
\end{aligned} \tag{B8}$$

$$\begin{aligned}
A_4^{\gamma\gamma'}(cnc) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4} g_1^{\gamma_j\gamma_{j+1}}\right]\delta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4}\delta_{\gamma_3\gamma_5}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1,\gamma_2}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})^3} + \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})^2}\right. \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^3(E_\gamma-E_{\gamma_2})} - \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma-E_{\gamma_1})^3(E_{\gamma_1}-E_{\gamma_2})} \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}}{(E_\gamma-E_{\gamma_2})^3(E_{\gamma_1}-E_{\gamma_2})} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})^2} \\
&\quad \left. -(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})} + \frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})}\right] \\
&\quad \times\left|g_1^{\gamma\gamma_1}\right|^2\left|g_1^{\gamma\gamma_2}\right|^2\eta_{\gamma_1\gamma_2}\delta_{\gamma\gamma'}.
\end{aligned} \tag{B9}$$

$$
\begin{aligned}
A_4^{\gamma\gamma'}(ncc) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4} g_1^{\gamma_j\gamma_{j+1}}\right]\eta_{\gamma_1\gamma_3}\delta_{\gamma_2\gamma_4}\delta_{\gamma_3\gamma_5}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma'})^2}+\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma'})^3}\right. \\
&\quad -\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma'})^2}-\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})^3} \\
&\quad -\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})^2}-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma'})^2} \\
&\quad \left.-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})^2}\right]\left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}. \qquad \text{(B10)}
\end{aligned}
$$

Then, we calculate the three terms with the single first contraction, that is, with one δ function. Because one δ function can not eliminate the whole apparent singularity, we also need to find out the nontrivial second contraction- and/or anti-contraction terms.

$$
\begin{aligned}
A_4^{\gamma\gamma'}(cnn,kc) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4} g_1^{\gamma_j\gamma_{j+1}}\right]\delta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4}\eta_{\gamma_3\gamma_5}\delta_{\gamma_2\gamma'}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1}\left[-\frac{2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})^3}-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma'})^2}\right. \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma'})^2}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})^2} \\
&\quad -\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^3(E_{\gamma_1}-E_{\gamma'})}+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})^2} \\
&\quad \left.-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})}\right]\left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}. \qquad \text{(B11)}
\end{aligned}
$$

$$
\begin{aligned}
A_4^{\gamma\gamma'}(cnn,kn) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4} g_1^{\gamma_j\gamma_{j+1}}\right]\delta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4}\eta_{\gamma_3\gamma_5}\eta_{\gamma_2\gamma'}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1,\gamma_2}\left[-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})^2}\right. \\
&\quad -\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})^2(E_\gamma-E_{\gamma'})} \\
&\quad -\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})} \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma'})} \\
&\quad -\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{(E_\gamma-E_{\gamma_2})^2(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_2}-E_{\gamma'})} \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})} \\
&\quad \left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})}\right] \\
&\quad \times|g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma'}\eta_{\gamma\gamma'}. \qquad \text{(B12)}
\end{aligned}
$$

$$
\begin{aligned}
A_4^{\gamma\gamma'}(ncn,c) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4}g_1^{\gamma_j\gamma_{j+1}}\right]\eta_{\gamma_1\gamma_3}\delta_{\gamma_2\gamma_4}\eta_{\gamma_3\gamma_5}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'}\delta_{\gamma\gamma'} \\
&= \sum_{\gamma_1,\gamma_2}\left[-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})^2}-\frac{2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^3(E_\gamma-E_{\gamma_2})}\right. \\
&\quad -\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma_2})^2}+\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^3(E_{\gamma_1}-E_{\gamma_2})} \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{(E_\gamma-E_{\gamma_2})^2(E_{\gamma_1}-E_{\gamma_2})^2}+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})} \\
&\quad \left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma_2})}\right]|g_1^{\gamma\gamma_1}|^2\,|g_1^{\gamma_1\gamma_2}|^2\,\eta_{\gamma\gamma_2}\delta_{\gamma\gamma'}. \qquad \text{(B13)}
\end{aligned}
$$

$$
\begin{aligned}
A_4^{\gamma\gamma'}(ncn,n) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4}g_1^{\gamma_j\gamma_{j+1}}\right]\eta_{\gamma_1\gamma_3}\delta_{\gamma_2\gamma_4}\eta_{\gamma_3\gamma_5}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'}\eta_{\gamma\gamma'} \\
&= \sum_{\gamma_1,\gamma_2}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})}\right. \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma'})^2} \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma_2})^2(E_{\gamma_1}-E_{\gamma'})} \\
&\quad -\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma'})} \\
&\quad -\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{(E_\gamma-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma_2})^2(E_{\gamma_2}-E_{\gamma'})} \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})^2(E_{\gamma_2}-E_{\gamma'})} \\
&\quad \left.-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma'})}\right] \\
&\quad \times|g_1^{\gamma_1\gamma_2}|^2\,g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma_2\gamma'}\eta_{\gamma\gamma'}. \qquad \text{(B14)}
\end{aligned}
$$

$$
\begin{aligned}
A_4^{\gamma\gamma'}(nnc,ck) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4}g_1^{\gamma_j\gamma_{j+1}}\right]\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4}\delta_{\gamma_3\gamma_5}\delta_{\gamma_1\gamma_4}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1}\left[-\frac{2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})^3}-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma'})^2}\right. \\
&\quad +\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma'})^2}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})^2} \\
&\quad -\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^3(E_{\gamma_1}-E_{\gamma'})}+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})^2} \\
&\quad \left.-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})}\right]\left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}. \qquad \text{(B15)}
\end{aligned}
$$

$$
\begin{aligned}
A_4^{\gamma\gamma'}(nnc,nk) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4}g_1^{\gamma_j\gamma_{j+1}}\right]\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_4}\delta_{\gamma_3\gamma_5}\eta_{\gamma_1\gamma_4}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1,\gamma_2}\left[\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})^2}\right. \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma'})^2} \\
&\quad+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{(E_\gamma-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_2}-E_{\gamma'})^2} \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})^2} \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})^2(E_{\gamma_2}-E_{\gamma'})} \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})} \\
&\quad\left.-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})}\right] \\
&\quad\times\left|g_1^{\gamma_2\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_2}\eta_{\gamma\gamma'}.
\end{aligned}
\tag{B16}
$$

Finally, we calculate the $A_4^{\gamma\gamma'}(nnn)$ by considering the two second decompositions, that is, its former three terms

$$
\begin{aligned}
A_4^{\gamma\gamma'}(nnn,cc) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4}g_1^{\gamma_j\gamma_{j+1}}\right]\left(\prod_{k=1}^{3}\eta_{\gamma_k\gamma_{k+2}}\right)\delta_{\gamma_1\gamma_4}\delta_{\gamma_2\gamma_5}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1}\left[-\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})^3}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma'})^2}\right. \\
&\quad+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma'})^2}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})^2} \\
&\quad-\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^3(E_{\gamma_1}-E_{\gamma'})}+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})^2} \\
&\quad\left.-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})}\right]g_1^{\gamma\gamma'}g_1^{\gamma'\gamma_1}g_1^{\gamma_1\gamma}g_1^{\gamma\gamma'}.
\end{aligned}
\tag{B17}
$$

$$
\begin{aligned}
A_4^{\gamma\gamma'}(nnn,cn) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4}g_1^{\gamma_j\gamma_{j+1}}\right]\left(\prod_{k=1}^{3}\eta_{\gamma_k\gamma_{k+2}}\right)\delta_{\gamma_1\gamma_4}\eta_{\gamma_2\gamma_5}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1,\gamma_2}\left[-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})^2}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})^2(E_\gamma-E_{\gamma'})}\right. \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})}+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma'})} \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{(E_\gamma-E_{\gamma_2})^2(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_2}-E_{\gamma'})}+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})} \\
&\quad\left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})}\right]g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma}g_1^{\gamma\gamma'}\eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}.
\end{aligned}
\tag{B18}
$$

$$
\begin{aligned}
A_4^{\gamma\gamma'}(nnn,nc) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4}g_1^{\gamma_j\gamma_{j+1}}\right]\left(\prod_{k=1}^{3}\eta_{\gamma_k\gamma_{k+2}}\right)\eta_{\gamma_1\gamma_4}\delta_{\gamma_2\gamma_5}\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'} \\
&= \sum_{\gamma_1,\gamma_2}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})^2}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma'})^2}\right. \\
&\quad+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{(E_\gamma-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_2}-E_{\gamma'})^2}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})^2} \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})^2(E_{\gamma_2}-E_{\gamma'})}-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})} \\
&\quad\left.-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})}\right]g_1^{\gamma\gamma'}g_1^{\gamma'\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}. \qquad \text{(B19)}
\end{aligned}
$$

while the fourth term has the third decomposition, that is

$$
\begin{aligned}
A_4^{\gamma\gamma'}(nnn,nn,c) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4}g_1^{\gamma_j\gamma_{j+1}}\right]\left(\prod_{k=1}^{3}\eta_{\gamma_k\gamma_{k+2}}\right)\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'}\delta_{\gamma\gamma'} \\
&= \sum_{\gamma_1,\gamma_2,\gamma_3}\left[-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma_3})^2}\right. \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})^2(E_\gamma-E_{\gamma_3})} \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma_3})} \\
&\quad+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma_3})} \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{(E_\gamma-E_{\gamma_2})^2(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_2}-E_{\gamma_3})} \\
&\quad+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_3}t}}{(E_\gamma-E_{\gamma_3})^2(E_{\gamma_1}-E_{\gamma_3})(E_{\gamma_2}-E_{\gamma_3})} \\
&\quad\left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma_3})}\right] \\
&\quad\times g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma_3}g_1^{\gamma_3\gamma}\eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_3}\delta_{\gamma\gamma'}. \qquad \text{(B20)}
\end{aligned}
$$

$$
\begin{aligned}
A_4^{\gamma\gamma'}(nnn,nn,n) &= \sum_{\gamma_1,\cdots,\gamma_5}\left[\sum_{i=1}^{5}(-1)^{i-1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}}{d_i(E[\gamma,4])}\right]\left[\prod_{j=1}^{4}g_1^{\gamma_j\gamma_{j+1}}\right]\left(\prod_{k=1}^{3}\eta_{\gamma_k\gamma_{k+2}}\right)\delta_{\gamma_1\gamma}\delta_{\gamma_5\gamma'}\eta_{\gamma\gamma'} \\
&= \sum_{\gamma_1,\gamma_2,\gamma_3}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma_3})(E_\gamma-E_{\gamma'})}\right. \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma_3})(E_{\gamma_1}-E_{\gamma'})} \\
&\quad+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{(E_\gamma-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_2}-E_{\gamma_3})(E_{\gamma_2}-E_{\gamma'})} \\
&\quad-\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_3}t}}{(E_\gamma-E_{\gamma_3})(E_{\gamma_1}-E_{\gamma_3})(E_{\gamma_2}-E_{\gamma_3})(E_{\gamma_3}-E_{\gamma'})} \\
&\quad\left.+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})(E_{\gamma_3}-E_{\gamma'})}\right] \\
&\quad\times g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma_3}g_1^{\gamma_3\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma\gamma'}\eta_{\gamma_1\gamma_3}\eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}. \qquad \text{(B21)}
\end{aligned}
$$

Now, all 15 contractions and/or anti-contractions in the fourth order approximation have been calculated out.

In order to add the contributions from the fourth order approximation to the improved forms of lower order perturbed solutions, we first decompose $A_4^{\gamma\gamma'}$, which is a summation of all above terms, into the three parts according to $\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}, (-it)\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}$ and $(-\mathrm{i}t)^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}/2$, that is

$$A_4^{\gamma\gamma'} = A_4^{\gamma\gamma'}(\mathrm{e}) + A_4^{\gamma\gamma'}(t\mathrm{e}) + A_4^{\gamma\gamma'}(t^2\mathrm{e}). \tag{B22}$$

Secondly, we decompose its every term into three parts according to $\mathrm{e}^{-\mathrm{i}E_\gamma t}, \mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}$ $(\sum_{\gamma_1}\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t})$ and $\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}$, that is

$$\begin{aligned}
A_4^{\gamma\gamma'}(\mathrm{e}) &= A_4^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_\gamma t}) + A_4^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}) + A_4^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}), &(B23)\\
A_4^{\gamma\gamma'}(t\mathrm{e}) &= A_4^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_\gamma t}) + A_4^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}) + A_4^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}), &(B24)\\
A_4^{\gamma\gamma'}(t^2\mathrm{e}) &= A_4^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t}) + A_4^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}) + A_4^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}). &(B25)
\end{aligned}$$

Finally, we again decompose every term in above equations into the diagonal and non-diagonal parts about γ and γ', that is

$$\begin{aligned}
A_4^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}) &= A_4^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t};\mathrm{D}) + A_4^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t};\mathrm{N}), &(B26)\\
A_4^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}) &= A_4^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t};\mathrm{D}) + A_4^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t};\mathrm{N}), &(B27)\\
A_4^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t}) &= A_4^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t};\mathrm{D}) + A_4^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i}t};\mathrm{N}). &(B28)
\end{aligned}$$

where E_{γ_i} takes E_γ, E_{γ_1} and $E_{\gamma'}$.

If we do not concern the improved forms of perturbed solutions equal to or more than the fourth order one, we only need to write down the second and third terms in eq.(B22) and calculate their diagonal and nondiagonal parts respectively. Based on the calculated results above, it is easy to obtain

$$\begin{aligned}
A_4^{\gamma\gamma'}\left(t\mathrm{e}^{-\mathrm{i}E_\gamma t};\mathrm{D}\right) &= (-\mathrm{i}t)\,\mathrm{e}^{-\mathrm{i}E_\gamma t}\left[\sum_{\gamma_1}\frac{-2\left|g_1^{\gamma\gamma_1}\right|^4}{\left(E_\gamma-E_{\gamma_1}\right)^3} - \sum_{\gamma_1,\gamma_2}\frac{\left|g_1^{\gamma\gamma_1}\right|^2\left|g_1^{\gamma\gamma_2}\right|^2\eta_{\gamma_1\gamma_2}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma_2}\right)^2}\right.\\
&\quad - \sum_{\gamma_1,\gamma_2}\frac{\left|g_1^{\gamma\gamma_1}\right|^2\left|g_1^{\gamma\gamma_2}\right|^2\eta_{\gamma_1\gamma_2}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_\gamma-E_{\gamma_2}\right)} + \sum_{\gamma_1,\gamma_2}\frac{\left|g_1^{\gamma\gamma_1}\right|^2\left|g_1^{\gamma_1\gamma_2}\right|^2\eta_{\gamma\gamma_2}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_\gamma-E_{\gamma_2}\right)}\\
&\quad \left. + \sum_{\gamma_1,\gamma_2,\gamma_3}\frac{g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma_3}g_1^{\gamma_3\gamma}\eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_3}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma_2}\right)\left(E_\gamma-E_{\gamma_3}\right)}\right]\delta_{\gamma\gamma'}. &(B29)
\end{aligned}$$

Substituting the relation $\eta_{\beta_1\beta_2} = 1-\delta_{\beta_1\beta_2}$, using the technology of index exchanging and introducing the definitions of so-called revision energy $G_\gamma^{(a)}$:

$$G_\gamma^{(2)} = \sum_{\gamma_1}\frac{\left|g_1^{\gamma\gamma_1}\right|^2}{E_\gamma-E_{\gamma_1}} \tag{B30}$$

$$G_\gamma^{(4)} = \sum_{\gamma_1,\gamma_2,\gamma_3}\frac{g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma_3}g_1^{\gamma_3\gamma}\eta_{\gamma\gamma_2}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma_2}\right)\left(E_\gamma-E_{\gamma_3}\right)} - \sum_{\gamma_1,\gamma_2}\frac{g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma}g_1^{\gamma\gamma_2}g_1^{\gamma_2\gamma}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_\gamma-E_{\gamma_2}\right)}, \tag{B31}$$

we can simplify eq.(refA4gammaD) to the following concise form:

$$A_4^{\gamma\gamma'}\left(t\mathrm{e}^{-\mathrm{i}E_\gamma t};\mathrm{D}\right) = -(-\mathrm{i}t)\,\mathrm{e}^{-\mathrm{i}E_\gamma t}\left[\sum_{\gamma_1}\frac{G_\gamma^{(2)}}{\left(E_\gamma-E_{\gamma_1}\right)^2}\left|g_1^{\gamma\gamma_1}\right|^2 - G_\gamma^{(4)}\right]\delta_{\gamma\gamma'}. \tag{B32}$$

Similar calculation and simplification lead to

$$A_4^{\gamma\gamma'}\left(t\mathrm{e}^{-\mathrm{i}E_\gamma t};\mathrm{N}\right) = (-\mathrm{i}t)\mathrm{e}^{-\mathrm{i}E_\gamma t}\left[\frac{G_\gamma^{(3)}g_1^{\gamma\gamma'}}{\left(E_\gamma-E_{\gamma'}\right)} + \sum_{\gamma_1}\frac{G_\gamma^{(2)}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma'}\right)}\right], \tag{B33}$$

where

$$G_{\gamma}^{(3)} = \sum_{\gamma_1,\gamma_2} \frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})}. \tag{B34}$$

For saving the space, the corresponding detail is omitted. In fact, it is not difficult, but it is necessary to be careful enough, specially in the cases of higher order approximations.

In the same way, we can obtain:

$$A_4^{\gamma\gamma'}\left(t\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t};\mathrm{D}\right) = (-\mathrm{i}t)\sum_{\gamma_1} \frac{G_{\gamma_1}^{(2)}\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma - E_{\gamma_1})^2} \left|g_1^{\gamma\gamma_1}\right|^2 \delta_{\gamma\gamma'}, \tag{B35}$$

$$A_4^{\gamma\gamma'}\left(t\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t};\mathrm{N}\right) = -(-\mathrm{i}t)\sum_{\gamma_1} \frac{G_{\gamma_1}^{(2)}\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}. \tag{B36}$$

$$A_4^{\gamma\gamma'}\left(t\mathrm{e}^{-\mathrm{i}E_{\gamma'}t};\mathrm{D}\right) = 0, \tag{B37}$$

$$\begin{aligned} A_4^{\gamma\gamma'}\left(t\mathrm{e}^{-\mathrm{i}E_{\gamma'}t};\mathrm{N}\right) &= -(-\mathrm{i}t)\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}\left[\sum_{\gamma_1} \frac{G_{\gamma'}^{(3)}}{(E_\gamma - E_{\gamma'})} g_1^{\gamma\gamma'} \right.\\ &\left. + \sum_{\gamma_1} \frac{G_{\gamma'}^{(2)}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}\right]. \end{aligned} \tag{B38}$$

For t^2e terms, only one term is nonzero, that is

$$A_4^{\gamma\gamma'}\left(t^2\mathrm{e}\right) = A_4^{\gamma\gamma'}\left(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t};D\right) = \frac{(-\mathrm{i}G_\gamma^{(2)}t)^2}{2!}\mathrm{e}^{-\mathrm{i}E_\gamma t}. \tag{B39}$$

since

$$A_4^{\gamma\gamma'}\left(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t};\mathrm{D}\right) = A_4^{\gamma\gamma'}\left(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t};\mathrm{D}\right) = 0, \tag{B40}$$

$$A_4^{\gamma\gamma'}\left(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t};\mathrm{N}\right) = A_4^{\gamma\gamma'}\left(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t};\mathrm{D}\right) = A_4^{\gamma\gamma'}\left(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t};\mathrm{D}\right) = 0. \tag{B41}$$

We can see that these terms can be merged with the lower order approximation to obtain the improved forms of perturbed solutions.

2. l=5 case

Now let we consider the case of the fifth order approximation ($l = 5$). From eq.(120) it follows that the first decompositions of g-product have $2^4 = 16$ terms. They can be divided into 5 group

$$A_5^{\gamma\gamma'} = \sum_{i=0}^{4} \mathcal{A}_5^{\gamma\gamma'}(i;\eta), \tag{B42}$$

where i indicates the number of η functions. Obviously

$$\mathcal{A}_5^{\gamma\gamma'}(0;\eta) = A_5^{\gamma\gamma'}(cccc), \tag{B43}$$

$$\mathcal{A}_5^{\gamma\gamma'}(1;\eta) = A_5^{\gamma\gamma'}(cccn) + A_5^{\gamma\gamma'}(ccnc) + A_5^{\gamma\gamma'}(cncc) + A_5^{\gamma\gamma'}(nccc), \tag{B44}$$

$$\begin{aligned} \mathcal{A}_5^{\gamma\gamma'}(2;\eta) &= A_5^{\gamma\gamma'}(ccnn) + A_5^{\gamma\gamma'}(cncn) + A_5^{\gamma\gamma'}(cnnc) \\ &+ A_5^{\gamma\gamma'}(nccn) + A_5^{\gamma\gamma'}(ncnc) + A_5^{\gamma\gamma'}(nncc), \end{aligned} \tag{B45}$$

$$\mathcal{A}_5^{\gamma\gamma'}(3;\eta) = A_5^{\gamma\gamma'}(cnnn) + A_5^{\gamma\gamma'}(ncnn) + A_5^{\gamma\gamma'}(nncn) + A_5^{\gamma\gamma'}(nnnc), \tag{B46}$$

$$\mathcal{A}_5^{\gamma\gamma'}(4;\eta) = A_5^{\gamma\gamma'}(nnnn). \tag{B47}$$

Here, we have used the notations stated in Sec. VI.

By calculation, we obtain the $\mathcal{A}_5^{\gamma\gamma'}(0,\eta)$ and every term of $\mathcal{A}_5^{\gamma\gamma'}(1,\eta)$ have only nontrivial first contractions and/or anti-contractions. But, we can find that every term of $\mathcal{A}_5^{\gamma\gamma'}(2,\eta)$ can have one nontrivial second or third or fourth contraction or anti-contraction, that is

$$A_5^{\gamma\gamma'}(ccnn) = A_5^{\gamma\gamma'}(ccnn,kkc) + A_5^{\gamma\gamma'}(ccnn,kkn), \tag{B48}$$
$$A_5^{\gamma\gamma'}(cncn) = A_5^{\gamma\gamma'}(cncn,kc) + A_5^{\gamma\gamma'}(cncn,kc), \tag{B49}$$
$$A_5^{\gamma\gamma'}(cnnc) = A_5^{\gamma\gamma'}(cnnc,kck) + A_5^{\gamma\gamma'}(cnnc,kck), \tag{B50}$$
$$A_5^{\gamma\gamma'}(nccn) = A_5^{\gamma\gamma'}(nccn,c) + A_5^{\gamma\gamma'}(nccn,n), \tag{B51}$$
$$A_5^{\gamma\gamma'}(ncnc) = A_5^{\gamma\gamma'}(ncnc,ck) + A_5^{\gamma\gamma'}(ncnc,nk), \tag{B52}$$
$$A_5^{\gamma\gamma'}(nncc) = A_5^{\gamma\gamma'}(nncc,ckk) + A_5^{\gamma\gamma'}(nncc,nkk). \tag{B53}$$

Similarly, every term of $\mathcal{A}_5^{\gamma\gamma'}(3,\eta)$ can have two higher order contractions and/or anti-contraction:

$$\begin{aligned} A_5^{\gamma\gamma'}(cnnn) &= A_5^{\gamma\gamma'}(cnnn,kcc) + A_5^{\gamma\gamma'}(cnnn,kcn) \\ &\quad + A_5^{\gamma\gamma'}(cnnn,knc) + A_5^{\gamma\gamma'}(cnnn,knn), \end{aligned} \tag{B54}$$
$$\begin{aligned} A_5^{\gamma\gamma'}(ncnn) &= A_5^{\gamma\gamma'}(ncnn,kkc,ck) + A_5^{\gamma\gamma'}(ncnn,kkn,ck) \\ &\quad + A_5^{\gamma\gamma'}(ncnn,kkc,nk) + A_5^{\gamma\gamma'}(ncnn,kkn,nk), \end{aligned} \tag{B55}$$
$$\begin{aligned} A_5^{\gamma\gamma'}(nncn) &= A_5^{\gamma\gamma'}(nncn,ckk,kc) + A_5^{\gamma\gamma'}(nncn,ckk,kn) \\ &\quad + A_5^{\gamma\gamma'}(nncn,nkk,kc) + A_5^{\gamma\gamma'}(nncn,nkk,kn), \end{aligned} \tag{B56}$$
$$\begin{aligned} A_5^{\gamma\gamma'}(nnnc) &= A_5^{\gamma\gamma'}(nnnc,cck) + A_5^{\gamma\gamma'}(nnnc,cnk) \\ &\quad + A_5^{\gamma\gamma'}(nnnc,nck) + A_5^{\gamma\gamma'}(nnnc,nnk). \end{aligned} \tag{B57}$$

Moreover, their last terms, with two higher order anti-contractions, can have one nontrivial more higher contraction or anti-contraction:

$$A_5^{\gamma\gamma'}(cnnn,knn) = A_5^{\gamma\gamma'}(cnnn,knn,kc) + A_5^{\gamma\gamma'}(cnnn,knn.kn), \tag{B58}$$
$$A_5^{\gamma\gamma'}(ncnn,kkn,nk) = A_5^{\gamma\gamma'}(ncnn,kkn,nk,c) + A_5^{\gamma\gamma'}(ncnn,kkn,nk,n), \tag{B59}$$
$$A_5^{\gamma\gamma'}(nncn,nkk,kn) = A_5^{\gamma\gamma'}(nncn,nkk,kn,c) + A_5^{\gamma\gamma'}(nncn,nkk,kn,n), \tag{B60}$$
$$A_5^{\gamma\gamma'}(nnnc,nnk) = A_5^{\gamma\gamma'}(nnnc,nnk,ck) + A_5^{\gamma\gamma'}(nnnc,nnk,nk). \tag{B61}$$

In the case of $A_5^{\gamma\gamma'}(nnnn)$, there are three second decompositions that result in

$$\begin{aligned} A_5^{\gamma\gamma'}(nnnn) &= A_5^{\gamma\gamma'}(nnnn,ccc) + A_5^{\gamma\gamma'}(nnnn,ccn) + A_5^{\gamma\gamma'}(nnnn,cnc) + A_5^{\gamma\gamma'}(nnnn,ncc) \\ &\quad + A_5^{\gamma\gamma'}(nnnn,cnn) + A_5^{\gamma\gamma'}(nnnn,ncn) + A_5^{\gamma\gamma'}(nnnn,nnc) + A_5^{\gamma\gamma'}(nnnn,nnn). \end{aligned} \tag{B62}$$

In the above expression, from the fifth term to the seventh term have the third- or fourth- contraction and anti-contraction, the eighth term has two third contractions and anti-contractions:

$$A_5^{\gamma\gamma'}(nnnn,cnn) = A_5^{\gamma\gamma'}(nnnn,cnn,kc) + A_5^{\gamma\gamma'}(nnnn,cnn,kn), \tag{B63}$$
$$A_5^{\gamma\gamma'}(nnnn,ncn) = A_5^{\gamma\gamma'}(nnnn,ncn,c) + A_5^{\gamma\gamma'}(nnnn,ncn,n), \tag{B64}$$
$$A_5^{\gamma\gamma'}(nnnn,nnc) = A_5^{\gamma\gamma'}(nnnn,nnc,ck) + A_5^{\gamma\gamma'}(nnnn,nnc,nk), \tag{B65}$$
$$\begin{aligned} A_5^{\gamma\gamma'}(nnnn,nnn) &= A_5^{\gamma\gamma'}(nnnn,nnn,cc) + A_5^{\gamma\gamma'}(nnnn,nnn,cn) \\ &\quad + A_5^{\gamma\gamma'}(nnnn,nnn,nc) + A_5^{\gamma\gamma'}(nnnn,nnn,nn). \end{aligned} \tag{B66}$$

In addition, $A_5^{\gamma\gamma'}(nnnn,nnn,nn)$ consists of the fourth contraction and anti-contraction

$$A_5^{\gamma\gamma'}(nnnn,nnn,nn) = A_5^{\gamma\gamma'}(nnnn,nnn,nn,c) + A_5^{\gamma\gamma'}(nnnn,nnn,nn,n). \tag{B67}$$

According to above analysis, we obtain the contribution from the five order approximation made of 52 terms after finding out all of contractions and anti-contractions.

Just like we have done in the $l=4$ case, we decompose

$$A_5^{\gamma\gamma'} = A_5^{\gamma\gamma'}(\mathrm{e}) + A_5^{\gamma\gamma'}(t\mathrm{e}) + A_5^{\gamma\gamma'}(t^2\mathrm{e}), \tag{B68}$$

where

$$\begin{aligned} A_5^{\gamma\gamma'}(\mathrm{e}) &= A_5^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_\gamma t}) + A_5^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}) + A_5^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}) + A_5^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}), &\text{(B69)}\\ A_4^{\gamma\gamma'}(t\mathrm{e}) &= A_5^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_\gamma t}) + A_5^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}) + A_5^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}) + A_5^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}), &\text{(B70)}\\ A_5^{\gamma\gamma'}(t^2\mathrm{e}) &= A_5^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t}) + A_5^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}) + A_5^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}) + A_5^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}). &\text{(B71)} \end{aligned}$$

While, every term in above equations has its diagonal and non-diagonal parts about γ and γ', that is

$$\begin{aligned} A_5^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t}) &= A_5^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{D}) + A_5^{\gamma\gamma'}(\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{N}), &\text{(B72)}\\ A_5^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t}) &= A_5^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{D}) + A_5^{\gamma\gamma'}(t\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{N}), &\text{(B73)}\\ A_5^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t}) &= A_5^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{D}) + A_5^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{N}). &\text{(B74)} \end{aligned}$$

where E_{γ_i} takes $E_\gamma, E_{\gamma_1}, E_{\gamma_2}$ and $E_{\gamma'}$.

If we do not concern the improved forms of perturbed solution more than the fourth order one, we only need to write down the second and third terms in eq.(B68). In the following, we respectively calculate them term by term, and put the second and third terms in eq.(B68) together as $A_5^{\gamma\gamma'}(t\mathrm{e},t^2\mathrm{e})$.

$$\begin{aligned} A_5^{\gamma\gamma'}(cccc;t\mathrm{e},t^2\mathrm{e}) &= \left[-(-\mathrm{i}t)\frac{3\mathrm{e}^{-\mathrm{i}E_\gamma t} + 3\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^4} + \frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma'})^3}\right.\\ &\quad \left. - \frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^3}\right]\left|g_1^{\gamma\gamma'}\right|^4 g_1^{\gamma\gamma'}. \end{aligned} \tag{B75}$$

$$\begin{aligned} A_5^{\gamma\gamma'}(cccn;t\mathrm{e},t^2\mathrm{e}) &= \sum_{\gamma_1}\left[-(-\mathrm{i}t)\frac{2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^3 (E_\gamma - E_{\gamma'})} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2}\right.\\ &\quad \left. -(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^3 (E_{\gamma_1} - E_{\gamma'})} + \frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})}\right]\\ &\quad \times |g_1^{\gamma\gamma_1}|^4 g_1^{\gamma\gamma'}\eta_{\gamma_1\gamma'}. \end{aligned} \tag{B76}$$

$$\begin{aligned} A_5^{\gamma\gamma'}(ccnc;t\mathrm{e},t^2\mathrm{e}) &= \sum_{\gamma_1}\left[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^3} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2}\right.\\ &\quad \left. -(-\mathrm{i}t)\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} - \frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})}\right]\\ &\quad \times \left|g_1^{\gamma_1\gamma'}\right|^2 \left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma'}\eta_{\gamma\gamma_1}. \end{aligned} \tag{B77}$$

$$\begin{aligned} A_5^{\gamma\gamma'}(cncc;t\mathrm{e},t^2\mathrm{e}) &= \sum_{\gamma_1}\left[-(-\mathrm{i}t)\frac{2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^3} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2}\right.\\ &\quad \left. +(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} + \frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^2}\right]\\ &\quad \times |g_1^{\gamma\gamma_1}|^2 \left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma'}\eta_{\gamma_1\gamma'}. \end{aligned} \tag{B78}$$

$$A_5^{\gamma\gamma'}(nccc;\mathrm{te},t^2\mathrm{e}) = \sum_{\gamma_1}\left[-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})^3} - (-\mathrm{i}t)\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})^3}\right.$$
$$\left.-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})^2} - \frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})^2}\right]$$
$$\times\left|g_1^{\gamma_1\gamma'}\right|^4 g_1^{\gamma\gamma'}\eta_{\gamma\gamma_1}. \tag{B79}$$

$$A_5^{\gamma\gamma'}(ccnn,kkc;\mathrm{te},t^2\mathrm{e})$$
$$= \sum_{\gamma_1,\gamma_2}\left[-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})^2} - (-\mathrm{i}t)\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})^3(E_\gamma - E_{\gamma_2})}\right.$$
$$\left.-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma - E_{\gamma_1})^3(E_{\gamma_1} - E_{\gamma_2})} + \frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})}\right]$$
$$\times|g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'}. \tag{B80}$$

$$A_5^{\gamma\gamma'}(ccnn,kkn;\mathrm{te},t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\left[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})}\right.$$
$$\left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma - E_{\gamma_1})^2(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})}\right]$$
$$\times|g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma_1\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}. \tag{B81}$$

$$A_5^{\gamma\gamma'}(cncn,kc;\mathrm{te},t^2\mathrm{e})$$
$$= \sum_{\gamma_1}\left[-(-\mathrm{i}t)\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^3} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma'})^2}\right.$$
$$\left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma - E_{\gamma'})^3(E_{\gamma_1} - E_{\gamma'})} + \frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^2}\right]$$
$$\times|g_1^{\gamma\gamma_1}|^2\left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma'}\eta_{\gamma_1\gamma'}. \tag{B82}$$

$$A_5^{\gamma\gamma'}(cncn,kn;\mathrm{te},t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\left[-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2}\right.$$
$$-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^2(E_\gamma - E_{\gamma'})}$$
$$-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})}$$
$$\left.+\frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})}\right]$$
$$\times|g_1^{\gamma\gamma_1}|^2|g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}. \tag{B83}$$

$$A_5^{\gamma\gamma'}(cnnc,kck;\mathrm{te},t^2\mathrm{e})$$
$$= \sum_{\gamma_1}\left[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma'})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma - E_{\gamma_1})^2(E_{\gamma_1} - E_{\gamma'})^2}\right.$$
$$\left.+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma - E_{\gamma'})^2(E_{\gamma_1} - E_{\gamma'})^2}\right]|g_1^{\gamma\gamma_1}|^2\left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'}. \tag{B84}$$

$$A_5^{\gamma\gamma'}(cnnc,knk;\mathrm{te},t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\Bigg[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})}\Bigg] \times |g_1^{\gamma\gamma_1}|^2\left|g_1^{\gamma_2\gamma'}\right|^2 g_1^{\gamma\gamma'}\eta_{\gamma_1\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma\gamma_2}. \quad \text{(B85)}$$

$$A_5^{\gamma\gamma'}(nccn,c;\mathrm{te},t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\Bigg[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma_2})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}}{(E_\gamma-E_{\gamma_2})^2(E_{\gamma_1}-E_{\gamma_2})^2}\Bigg]|g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'}. \quad \text{(B86)}$$

$$A_5^{\gamma\gamma'}(nccn,n;\mathrm{te},t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\Bigg[-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma_2})^2(E_{\gamma_1}-E_{\gamma'})} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}}{(E_\gamma-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma_2})^2(E_{\gamma_2}-E_{\gamma'})}\Bigg] \times |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma_1\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}. \quad \text{(B87)}$$

$$A_5^{\gamma\gamma'}(ncnc,ck;\mathrm{te},t^2\mathrm{e}) = \sum_{\gamma_1}\Bigg[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})^3} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})^2} - (-\mathrm{i}t)\frac{2\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})^3(E_{\gamma_1}-E_{\gamma'})} - \frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})}\Bigg] \times \left|g_1^{\gamma_1\gamma'}\right|^2\left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma'}\eta_{\gamma\gamma_1}. \quad \text{(B88)}$$

$$A_5^{\gamma\gamma'}(ncnc,nk;\mathrm{te},t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\Bigg[-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})^2} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})^2(E_{\gamma_2}-E_{\gamma'})} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})} - \frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})}\Bigg] \times \left|g_1^{\gamma_1\gamma'}\right|^2\left|g_1^{\gamma_2\gamma'}\right|^2 g_1^{\gamma\gamma'}\eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_2}. \quad \text{(B89)}$$

$$A_5^{\gamma\gamma'}(nncc,ckk;\mathrm{te},t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\Bigg[-(-\mathrm{i}t)\frac{2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})^3} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}}{(E_\gamma-E_{\gamma_2})^3(E_{\gamma_1}-E_{\gamma_2})} + \frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})^2}\Bigg] \times |g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'}. \quad \text{(B90)}$$

$$A_5^{\gamma\gamma'}(nncc,nkk;te,t^2e) = \sum_{\gamma_1,\gamma_2}\left[(-it)\frac{e^{-iE_{\gamma_2}t}}{(E_\gamma-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_2}-E_{\gamma'})^2}\right.$$
$$\left.+(-it)\frac{e^{-iE_{\gamma'}t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})^2}\right]$$
$$\times\left|g_1^{\gamma_2\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma_1\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}. \tag{B91}$$

$$A_5^{\gamma\gamma'}(cnnn,kcc;te,t^2e)$$
$$=\sum_{\gamma_1,\gamma_2}\left[-(-it)\frac{2e^{-iE_\gamma t}}{(E_\gamma-E_{\gamma_1})^3(E_\gamma-E_{\gamma_2})}-(-it)\frac{e^{-iE_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})^2}\right.$$
$$\left.-(-it)\frac{e^{-iE_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^3(E_{\gamma_1}-E_{\gamma_2})}+\frac{(-it)^2}{2!}\frac{e^{-iE_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})}\right]$$
$$\times|g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2}g_1^{\gamma_2\gamma_1}g_1^{\gamma_1\gamma}\delta_{\gamma\gamma'}. \tag{B92}$$

$$A_5^{\gamma\gamma'}(cnnn,kcn;te,t^2e) = \sum_{\gamma_1,\gamma_2}\left[(-it)\frac{e^{-iE_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})}\right.$$
$$\left.+(-it)\frac{e^{-iE_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2(E_{\gamma_1}-E_{\gamma_2})(E_{\gamma_1}-E_{\gamma'})}\right]$$
$$\times|g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2}g_1^{\gamma_2\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}\eta_{\gamma\gamma'}. \tag{B93}$$

$$A_5^{\gamma\gamma'}(cnnn,knc;te,t^2e) = \sum_{\gamma_1,\gamma_2,\gamma_3}\left[-(-it)\frac{e^{-iE_\gamma t}}{(E_\gamma-E_{\gamma_1})^2(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma_3})}\right.$$
$$-(-it)\frac{e^{-iE_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})^2(E_\gamma-E_{\gamma_3})}$$
$$-(-it)\frac{e^{-iE_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma_3})^2}$$
$$\left.+\frac{(-it)^2}{2!}\frac{e^{-iE_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma_3})}\right]$$
$$\times|g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2}g_1^{\gamma_2\gamma_3}g_1^{\gamma_3\gamma}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma_3}\delta_{\gamma\gamma'}. \tag{B94}$$

$$A_5^{\gamma\gamma'}(cnnn,knn,kc;te,t^2e) = \sum_{\gamma_1,\gamma_2}\left[(-it)\frac{e^{-iE_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma'})^2}\right.$$
$$\left.+(-it)\frac{e^{-iE_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})}\right]$$
$$\times\left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma'}. \tag{B95}$$

$$A_5^{\gamma\gamma'}(cnnn,knn,kn;te,t^2e)$$
$$=\sum_{\gamma_1,\gamma_2,\gamma_3}(-it)\frac{e^{-iE_\gamma t}|g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2}g_1^{\gamma_2\gamma_3}g_1^{\gamma_3\gamma'}\eta_{\gamma\gamma_3}\eta_{\gamma\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma_3}\eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma_3})(E_\gamma-E_{\gamma'})}. \tag{B96}$$

$$A_5^{\gamma\gamma'}(ncnn, kkc, ck; \mathrm{te}, t^2\mathrm{e})$$
$$= \sum_{\gamma_1}\left[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2}\right] |g_1^{\gamma\gamma_1}|^2 \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'}. \tag{B97}$$

$$A_5^{\gamma\gamma'}(ncnn, kkc, nk; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\left[-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})}\right] \times \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}. \tag{B98}$$

$$A_5^{\gamma\gamma'}(ncnn, kkn, ck; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\left[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})}\right] \times |g_1^{\gamma\gamma_1}|^2 |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma'} \eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}. \tag{B99}$$

$$A_5^{\gamma\gamma'}(ncnn, kkn, nk, c; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3}\left[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_3})}\right] \times |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma\gamma_2}\eta_{\gamma_2\gamma_3}\delta_{\gamma\gamma'}. \tag{B100}$$

$$A_5^{\gamma\gamma'}(ncnn, kkn, nk, n; \mathrm{te}, t^2\mathrm{e})$$
$$= -\sum_{\gamma_1,\gamma_2,\gamma_3}(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t} |g_1^{\gamma_1\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_3} g_1^{\gamma_3\gamma'} \eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma\gamma'}\eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma_3}\eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_3})(E_{\gamma_1} - E_{\gamma'})}. \tag{B101}$$

$$A_5^{\gamma\gamma'}(nncn, ckk; kc, \mathrm{te}, t^2\mathrm{e})$$
$$= \sum_{\gamma_1}\left[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} + (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2}\right] |g_1^{\gamma\gamma_1}|^2 \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'}. \tag{B102}$$

$$A_5^{\gamma\gamma'}(nncn, ckk, kn; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\left[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^2 (E_\gamma - E_{\gamma'})} - (-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}}{(E_\gamma - E_{\gamma_2})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma'})}\right] \times |g_1^{\gamma\gamma_2}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma'}\eta_{\gamma_1\gamma'}. \tag{B103}$$

$$A_5^{\gamma\gamma'}(nncn, nkk, kc; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\Bigg[-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})^2}$$
$$+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})^2(E_{\gamma_2} - E_{\gamma'})}\Bigg]$$
$$\times |g_1^{\gamma_1\gamma_2}|^2 \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'}\eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}\eta_{\gamma_2\gamma'}. \quad \text{(B104)}$$

$$A_5^{\gamma\gamma'}(nncn, nkk, kn, c; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2\gamma_3}\Bigg[(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^2(E_\gamma - E_{\gamma_3})}$$
$$-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}}{(E_\gamma - E_{\gamma_2})^2(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma_3})}\Bigg]$$
$$\times |g_1^{\gamma_2\gamma_3}|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma}\eta_{\gamma\gamma_3}\eta_{\gamma_1\gamma_3}\delta_{\gamma\gamma'}. \quad \text{(B105)}$$

$$A_5^{\gamma\gamma'}(nncn, nkk, kn, n; \mathrm{te}, t^2\mathrm{e})$$
$$= \sum_{\gamma_1,\gamma_2,\gamma_3}(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t}|g_1^{\gamma_2\gamma_3}|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma\gamma'}\eta_{\gamma_1\gamma_3}\eta_{\gamma_1\gamma'}\eta_{\gamma_3\gamma'}}{(E_\gamma - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma_3})(E_{\gamma_2} - E_{\gamma'})}. \quad \text{(B106)}$$

$$A_5^{\gamma\gamma'}(nnnc, cck; \mathrm{te}, t^2\mathrm{e})$$
$$= \sum_{\gamma_1,\gamma_2}\Bigg[-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})^2} - (-\mathrm{i}t)\frac{2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^3(E_\gamma - E_{\gamma_2})}$$
$$-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma - E_{\gamma_1})^3(E_{\gamma_1} - E_{\gamma_2})} + \frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})}\Bigg]$$
$$\times |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'}. \quad \text{(B107)}$$

$$A_5^{\gamma\gamma'}(nnnc, cnk; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3}\Bigg[-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})^2}$$
$$-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^2(E_\gamma - E_{\gamma_3})}$$
$$-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})}$$
$$+\frac{(-\mathrm{i}t)^2}{2!}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})}\Bigg]$$
$$\times |g_1^{\gamma\gamma_3}|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma}\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_3}\delta_{\gamma\gamma'}. \quad \text{(B108)}$$

$$A_5^{\gamma\gamma'}(nnnc, nck; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2}\Bigg[-(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})^2}$$
$$+(-\mathrm{i}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})^2(E_{\gamma_2} - E_{\gamma'})}\Bigg]$$
$$\times \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}. \quad \text{(B109)}$$

$$
\begin{aligned}
A_5^{\gamma\gamma'}(nnnc, nnk, ck; \mathrm{te}, t^2\mathrm{e}) \; = \; & \sum_{\gamma_1,\gamma_2} \left[(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2} \right. \\
& \left. +(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})} \right] \\
& \times \left| g_1^{\gamma\gamma'} \right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma'}. \qquad \text{(B110)}
\end{aligned}
$$

$$
\begin{aligned}
& A_5^{\gamma\gamma'}(nnnc, nnk, nk; \mathrm{te}, t^2\mathrm{e}) \\
& = \sum_{\gamma_1,\gamma_2,\gamma_3} (-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t} \left| g_1^{\gamma_3\gamma'} \right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma_3} \eta_{\gamma\gamma'} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma_3}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})(E_{\gamma_3} - E_{\gamma'})}. \qquad \text{(B111)}
\end{aligned}
$$

$$
\begin{aligned}
& A_5^{\gamma\gamma'}(nnnn, ccc; \mathrm{te}, t^2\mathrm{e}) \\
& = \sum_{\gamma_1} \left[(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2} + (-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right. \\
& \left. +(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})^2} \right] \left| g_1^{\gamma\gamma_1} \right|^2 \left(g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \right)^2 g_1^{\gamma\gamma'}. \qquad \text{(B112)}
\end{aligned}
$$

$$
\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, ccn; \mathrm{te}, t^2\mathrm{e}) \; = \; & \sum_{\gamma_1,\gamma_2} \left[(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})} \right. \\
& \left. +(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})} \right] \\
& \times \left(g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} \right) g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'} \eta_{\gamma_2\gamma'}. \qquad \text{(B113)}
\end{aligned}
$$

$$
\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, cnc; \mathrm{te}, t^2\mathrm{e}) \; = \; & \sum_{\gamma_1,\gamma_2} \left[(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2} \right. \\
& \left. +(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})} \right] \\
& \times \left(g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \right) \left(g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \right) g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_2}. \qquad \text{(B114)}
\end{aligned}
$$

$$
\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, ncc; \mathrm{te}, t^2\mathrm{e}) \; = \; & \sum_{\gamma_1,\gamma_2} \left[-(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})^2} \right. \\
& \left. +(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})^2 (E_{\gamma_2} - E_{\gamma'})} \right] \\
& \times \left(g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma_1} g_1^{\gamma_1\gamma'} \right) g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2} \eta_{\gamma\gamma'}. \qquad \text{(B115)}
\end{aligned}
$$

$$
\begin{aligned}
A_5^{\gamma\gamma'}(nnnn, cnn, kc; \mathrm{te}, t^2\mathrm{e}) \; = \; & \sum_{\gamma_1,\gamma_2} \left[(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2} \right. \\
& \left. +(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})} \right] \\
& \times \left(g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} \right) \left(g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \right) g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_2}. \qquad \text{(B116)}
\end{aligned}
$$

$$A_5^{\gamma\gamma'}(nnnn, cnn, kn; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3} (-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma_3} g_1^{\gamma_3\gamma'} \eta_{\gamma\gamma'}\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_3}\eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma'})}. \tag{B117}$$

$$\begin{aligned} A_5^{\gamma\gamma'}(nnnn, ncn, c; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2\gamma_3} \Bigg[&(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})} \\ &+(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2 (E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_3})} \Bigg] \\ &\times |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_1} \eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\delta_{\gamma\gamma'}. \end{aligned} \tag{B118}$$

$$A_5^{\gamma\gamma'}(nnnn, ncn, n; \mathrm{te}, t^2\mathrm{e}) = -\sum_{\gamma_1,\gamma_2,\gamma_3} (-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma_2\gamma'}\eta_{\gamma_3\gamma'}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_3})(E_{\gamma_1} - E_{\gamma'})}. \tag{B119}$$

$$\begin{aligned} A_5^{\gamma\gamma'}(nnnn, nnc, ck; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2} \Bigg[&(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2} \\ &+(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})} \Bigg] \\ &\times \left(g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}\right) \left(g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma}\right) g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_2}. \end{aligned} \tag{B120}$$

$$A_5^{\gamma\gamma'}(nnnn, nnc, nk; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3} (-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t} g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma'} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma_3}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})(E_{\gamma_3} - E_{\gamma'})}. \tag{B121}$$

$$\begin{aligned} A_5^{\gamma\gamma'}(nnnn, nnn, cc; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2} \Bigg[&(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2} \\ &+(-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^2 (E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})} \Bigg] \\ &\times g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma\gamma_1}\eta_{\gamma_2\gamma'}. \end{aligned} \tag{B122}$$

$$A_5^{\gamma\gamma'}(nnnn, nnn, cn; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3} (-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} g_1^{\gamma\gamma'} \eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_3}\eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}\eta_{\gamma_3\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma'})}. \tag{B123}$$

$$A_5^{\gamma\gamma'}(nnnn, nnn, nc; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3} (-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma'} g_1^{\gamma\gamma'} \eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})(E_{\gamma_3} - E_{\gamma'})}. \tag{B124}$$

$$A_5^{\gamma\gamma'}(nnnn, nnn, nn, c; \mathrm{te}, t^2\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} (-\mathrm{i}t) \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4} g_1^{\gamma_4\gamma} \eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma_1\gamma_3}\eta_{\gamma_1\gamma_4}\eta_{\gamma_2\gamma_4}\delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma_4})}. \tag{B125}$$

$$A_5^{\gamma\gamma'}(nnnn, nnn, nn, n; t\mathrm{e}, t^2\mathrm{e}) \;=\; 0 \tag{B126}$$

Based on above 52 contraction- and anti contraction- expressions, we can, via the reorganization and summation, obtain

$$\begin{aligned} A_5(t\mathrm{e}^{-\mathrm{i}E_\gamma t}, \mathrm{D}) \;=\;& -(-\mathrm{i}G_\gamma^{(3)}t)\sum_{\gamma_1}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma}\delta_{\gamma\gamma'} \\ & -(-\mathrm{i}G_\gamma^{(2)}t)\sum_{\gamma_1,\gamma_2}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})}\right. \\ & \left. +\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^2}\right] g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'} + (-\mathrm{i}G_\gamma^{(5)}t)\delta_{\gamma\gamma'} \end{aligned} \tag{B127}$$

where

$$\begin{aligned} G_\gamma^{(5)} \;=\;& \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\frac{g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4} g_1^{\gamma_4\gamma}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma_4})} \\ & - \sum_{\gamma_1,\gamma_2,\gamma_3}\left[\frac{g_1^{\gamma\gamma_1} g_1^{\gamma\gamma_2} g_1^{\gamma_1\gamma} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})} + \frac{g_1^{\gamma\gamma_1} g_1^{\gamma\gamma_2} g_1^{\gamma_1\gamma} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})^2(E_\gamma - E_{\gamma_3})}\right. \\ & \left. +\frac{g_1^{\gamma\gamma_1} g_1^{\gamma\gamma_2} g_1^{\gamma_1\gamma} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})^2}\right]. \end{aligned} \tag{B128}$$

$$\begin{aligned} A_5(t\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}, \mathrm{D}) \;=\;& (-\mathrm{i}G_\gamma^{(3)}t)\sum_{\gamma_1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma}\delta_{\gamma\gamma'} \\ & +(-\mathrm{i}G_\gamma^{(2)}t)\sum_{\gamma_1,\gamma_2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma - E_{\gamma_1})^2(E_{\gamma_1} - E_{\gamma_2})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'} \end{aligned} \tag{B129}$$

$$A_5(t\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}, \mathrm{D}) \;=\; -(-\mathrm{i}G_\gamma^{(2)}t)\sum_{\gamma_1,\gamma_2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}}{(E_\gamma - E_{\gamma_2})^2(E_{\gamma_1} - E_{\gamma_2})} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}\delta_{\gamma\gamma'} \tag{B130}$$

$$A_5(t\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}, \mathrm{D}) \;=\; 0 \tag{B131}$$

$$\begin{aligned} A_5(t\mathrm{e}^{-\mathrm{i}E_\gamma t}, \mathrm{N}) \;=\;& (-\mathrm{i}G_\gamma^{(4)}t)\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma'}}{(E_\gamma - E_{\gamma'})} + (-\mathrm{i}G_\gamma^{(3)}t)\sum_{\gamma_1}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}\eta^{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})} \\ & -(-\mathrm{i}G_\gamma^{(2)}t)\sum_{\gamma_1}\left[\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2(E_\gamma - E_{\gamma'})} + \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^2}\right] \\ & +(-\mathrm{i}G_\gamma^{(2)}t)\sum_{\gamma_1,\gamma_2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})} \end{aligned} \tag{B132}$$

$$\begin{aligned} A_5(t\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}, \mathrm{N}) \;=\;& -(-\mathrm{i}t)\sum_{\gamma_1}\frac{G_{\gamma_1}^{(3)}\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma'})} \\ & -(-\mathrm{i}t)\sum_{\gamma_1,\gamma_2}\frac{G_{\gamma_1}^{(2)}\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'}\eta_{\gamma_1\gamma'}\eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma'})} \end{aligned} \tag{B133}$$

$$A_5(t\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t}, \mathrm{N}) \;=\; (-\mathrm{i}t)\sum_{\gamma_1,\gamma_2}\frac{G_{\gamma_2}^{(2)}\mathrm{e}^{-\mathrm{i}E_{\gamma_2} t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_2})(E_{\gamma_1} - E_{\gamma_2})(E_{\gamma_2} - E_{\gamma'})} \tag{B134}$$

$$
\begin{aligned}
A_5(t\mathrm{e}^{-\mathrm{i}E_{\gamma'}t},\mathrm{N}) &= -(-\mathrm{i}G_{\gamma'}^{(4)}t)\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}g_1^{\gamma\gamma'}}{(E_\gamma-E_{\gamma'})}+(-\mathrm{i}G_{\gamma'}^{(3)}t)\sum_{\gamma_1}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta^{\gamma\gamma'}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})} \\
&\quad +(-\mathrm{i}G_{\gamma'}^{(2)}t)\sum_{\gamma_1}\left[\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}g_1^{\gamma'\gamma_1}g_1^{\gamma_1\gamma'}g_1^{\gamma\gamma'}}{(E_\gamma-E_{\gamma'})^2(E_{\gamma_1}-E_{\gamma'})}+\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}g_1^{\gamma'\gamma_1}g_1^{\gamma_1\gamma'}g_1^{\gamma\gamma'}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})^2}\right] \\
&\quad -(-\mathrm{i}G_{\gamma'}^{(2)}t)\sum_{\gamma_1,\gamma_2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma_1\gamma'}\eta_{\gamma\gamma'}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})(E_{\gamma_2}-E_{\gamma'})} \qquad \text{(B135)}
\end{aligned}
$$

For the parts with $t^2\mathrm{e}$, we have

$$
\begin{aligned}
A_5(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{D}) &= \frac{(-\mathrm{i}t)^2}{2!}2G_\gamma^{(2)}G_\gamma^{(3)}\delta_{\gamma\gamma'}\mathrm{e}^{-\mathrm{i}E_\gamma t}, \qquad \text{(B136)} \\
A_5(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t},\mathrm{D}) &= A_5(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t},\mathrm{D}) = A_5(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t},\mathrm{D}) = 0. \qquad \text{(B137)}
\end{aligned}
$$

$$
\begin{aligned}
A_5(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{N}) &= \frac{(-\mathrm{i}t)^2}{2!}\left(G_\gamma^{(2)}\right)^2\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{E_\gamma-E_{\gamma'}}g_1^{\gamma\gamma'}, \qquad \text{(B138)} \\
A_5(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_2}t},\mathrm{N}) &= A_5(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t},\mathrm{N}) = 0, \qquad \text{(B139)} \\
A_5(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'}t},\mathrm{N}) &= -\frac{(-\mathrm{i}t)^2}{2!}\left(G_{\gamma'}^{(2)}\right)^2\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{E_\gamma-E_{\gamma'}}g_1^{\gamma\gamma'}. \qquad \text{(B140)}
\end{aligned}
$$

It is clear that above diagonal and nondiagonal part about $A_5^{\gamma\gamma'}(te)$ and $A_5^{\gamma\gamma'}(te)$ indeed has the expected forms and can be merged reasonably to the lower order terms in order to obtain the improved forms of perturbed solutions.

3. $l = 6$ case

Now let we consider the case of the sixth order approximation ($l = 6$). From eq.(120) it follows that the first decompositions of g-product have $2^5 = 32$ terms. Like the $l = 5$ case, they can be divided into 6 group

$$
A_6^{\gamma\gamma'} = \sum_{i=0}^{4}\mathcal{A}_6^{\gamma\gamma'}(i;\eta), \qquad \text{(B141)}
$$

where i indicates the number of η functions. Obviously

$$
\mathcal{A}_6^{\gamma\gamma'}(0;\eta) = A_6^{\gamma\gamma'}(ccccc), \qquad \text{(B142)}
$$

$$
\begin{aligned}
\mathcal{A}_6^{\gamma\gamma'}(1;\eta) &= A_6^{\gamma\gamma'}(ccccn)+A_6^{\gamma\gamma'}(cccnc)+A_6^{\gamma\gamma'}(ccncc) \\
&\quad +A_6^{\gamma\gamma'}(cnccc)+A_6^{\gamma\gamma'}(ncccc), \qquad \text{(B143)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_6^{\gamma\gamma'}(2;\eta) &= A_6^{\gamma\gamma'}(cccn)+A_6^{\gamma\gamma'}(ccncn)+A_6^{\gamma\gamma'}(cnccn)+A_6^{\gamma\gamma'}(ncccn) \\
&\quad +A_6^{\gamma\gamma'}(ccnnc)+A_6^{\gamma\gamma'}(cncnc)+A_6^{\gamma\gamma'}(nccnc)+A_6^{\gamma\gamma'}(cnncc) \\
&\quad +A_6^{\gamma\gamma'}(ncncc)+A_6^{\gamma\gamma'}(nnccc), \qquad \text{(B144)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_6^{\gamma\gamma'}(3;\eta) &= A_6^{\gamma\gamma'}(ccnnn)+A_6^{\gamma\gamma'}(cncnn)+A_6^{\gamma\gamma'}(cnncn)+A_6^{\gamma\gamma'}(cnnnc) \\
&\quad +A_6^{\gamma\gamma'}(nccnn)+A_6^{\gamma\gamma'}(ncncn)+A_6^{\gamma\gamma'}(ncnnc)+A_6^{\gamma\gamma'}(nnccn) \\
&\quad +A_6^{\gamma\gamma'}(nncnc)+A_6^{\gamma\gamma'}(nnncc), \qquad \text{(B145)}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{A}_6^{\gamma\gamma'}(4;\eta) &= A_6^{\gamma\gamma'}(cnnnn)+A_6^{\gamma\gamma'}(ncnnn)+A_6^{\gamma\gamma'}(nncnn) \\
&\quad +A_6^{\gamma\gamma'}(nnncn)+A_6^{\gamma\gamma'}(nnnnc) \qquad \text{(B146)}
\end{aligned}
$$

$$\mathcal{A}_6^{\gamma\gamma'}(5;\eta) = A_6^{\gamma\gamma'}(nnnnn). \tag{B147}$$

Furthermore considering the high order contraction or anti-contraction, we have

$$\begin{aligned}
A_6^{\gamma\gamma'}(cccnn) &= A_6^{\gamma\gamma'}(cccnn,kkkc) + A_6^{\gamma\gamma'}(cccnn,kkkn), &\text{(B148)}\\
A_6^{\gamma\gamma'}(ccncn) &= A_6^{\gamma\gamma'}(ccncn,kkc) + A_6^{\gamma\gamma'}(ccncn,kkn), &\text{(B149)}\\
A_6^{\gamma\gamma'}(ccnnc) &= A_6^{\gamma\gamma'}(ccnnc,kkck) + A_6^{\gamma\gamma'}(ccnnc,kknk), &\text{(B150)}\\
A_6^{\gamma\gamma'}(cnccn) &= A_6^{\gamma\gamma'}(cnccn,kc) + A_6^{\gamma\gamma'}(cnccn,kn), &\text{(B151)}\\
A_6^{\gamma\gamma'}(cncnc) &= A_6^{\gamma\gamma'}(cncnc,kck) + A_6^{\gamma\gamma'}(cncnc,knk), &\text{(B152)}\\
A_6^{\gamma\gamma'}(cnncc) &= A_6^{\gamma\gamma'}(cnncc,kckk) + A_6^{\gamma\gamma'}(cnncc,knkk), &\text{(B153)}\\
A_6^{\gamma\gamma'}(ncccn) &= A_6^{\gamma\gamma'}(ncccn,c) + A_6^{\gamma\gamma'}(ncccn,n), &\text{(B154)}\\
A_6^{\gamma\gamma'}(nccnc) &= A_6^{\gamma\gamma'}(nccnc,ck) + A_6^{\gamma\gamma'}(nccnc,nk), &\text{(B155)}\\
A_6^{\gamma\gamma'}(ncncc) &= A_6^{\gamma\gamma'}(ncncc,ckk) + A_6^{\gamma\gamma'}(ncncc,nkk), &\text{(B156)}\\
A_6^{\gamma\gamma'}(nnccc) &= A_6^{\gamma\gamma'}(nnccc,ckkk) + A_6^{\gamma\gamma'}(nnccc,nkkk) &\text{(B157)}
\end{aligned}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(ccnnn) &= A_6^{\gamma\gamma'}(ccnnn,kkcc) + A_6^{\gamma\gamma'}(ccnnn,kkcn) + A_6^{\gamma\gamma'}(ccnnn,kknc)\\
&\quad + A_6^{\gamma\gamma'}(ccnnn,kknn,kkc) + A_6^{\gamma\gamma'}(ccnnn,kknn,kkn), &\text{(B158)}\\
A_6^{\gamma\gamma'}(cncnn) &= A_6^{\gamma\gamma'}(cncnn,kkkc,kck) + A_6^{\gamma\gamma'}(cncnn,kkkc,knk) + A_6^{\gamma\gamma'}(cncnn,kkkn,kck)\\
&\quad + A_6^{\gamma\gamma'}(cncnn,kkkn,knk,kc) + A_6^{\gamma\gamma'}(cncnn,kkkn,knk,kn), &\text{(B159)}\\
A_6^{\gamma\gamma'}(cnncn) &= A_6^{\gamma\gamma'}(cnncn,kckk,kkc) + A_6^{\gamma\gamma'}(cnncn,kckk,kkn) + A_6^{\gamma\gamma'}(cnncn,knkk,kkc)\\
&\quad + A_6^{\gamma\gamma'}(cnncn,knkk,kkn,kc) + A_6^{\gamma\gamma'}(cnncn,knkk,kkn,kn), &\text{(B160)}\\
A_6^{\gamma\gamma'}(cnnnc) &= A_6^{\gamma\gamma'}(cnnnc,kcck) + A_6^{\gamma\gamma'}(cnnnc,kcnk) + A_6^{\gamma\gamma'}(cnnnc,knck)\\
&\quad + A_6^{\gamma\gamma'}(cnnnc,knnk,kck) + A_6^{\gamma\gamma'}(cnnnc,knnk,knk), &\text{(B161)}\\
A_6^{\gamma\gamma'}(nccnn) &= A_6^{\gamma\gamma'}(nccnn,kkkc,ck) + A_6^{\gamma\gamma'}(nccnn,kkkc,nk) + A_6^{\gamma\gamma'}(nccnn,kkkn,ck)\\
&\quad + A_6^{\gamma\gamma'}(nccnn,kkkn,nk,c) + A_6^{\gamma\gamma'}(nccnn,kkkn,nk,n), &\text{(B162)}\\
A_6^{\gamma\gamma'}(ncncn) &= A_6^{\gamma\gamma'}(ncncn,ckc) + A_6^{\gamma\gamma'}(ncncn,ckn) + A_6^{\gamma\gamma'}(ncncn,nkc)\\
&\quad + A_6^{\gamma\gamma'}(ncncn,nkn,c) + A_6^{\gamma\gamma'}(ncncn,nkn,n), &\text{(B163)}\\
A_6^{\gamma\gamma'}(ncnnc) &= A_6^{\gamma\gamma'}(ncnnc,kkck,ckk) + A_6^{\gamma\gamma'}(ncnnc,kkck,nkk) + A_6^{\gamma\gamma'}(ncnnc,kknk,ckk)\\
&\quad + A_6^{\gamma\gamma'}(ncnnc,kknk,nkk,ck) + A_6^{\gamma\gamma'}(ncnnc,kknk,nkk,nk), &\text{(B164)}\\
A_6^{\gamma\gamma'}(nnccn) &= A_6^{\gamma\gamma'}(nnccn,cc) + A_6^{\gamma\gamma'}(nnccn,cn) + A_6^{\gamma\gamma'}(nnccn,nc)\\
&\quad + A_6^{\gamma\gamma'}(nnccn,nn,c) + A_6^{\gamma\gamma'}(nnccn,nn,n), &\text{(B165)}\\
A_6^{\gamma\gamma'}(nncnc) &= A_6^{\gamma\gamma'}(nncnc,ckkk,kck) + A_6^{\gamma\gamma'}(nncnc,ckkk,knk) + A_6^{\gamma\gamma'}(nncnc,nkkk,kck)\\
&\quad + A_6^{\gamma\gamma'}(nncnc,nkkk,knk,ck) + A_6^{\gamma\gamma'}(nncnc,nkkk,knk,nk), &\text{(B166)}\\
A_6^{\gamma\gamma'}(nnncc) &= A_6^{\gamma\gamma'}(nnncc,cckk) + A_6^{\gamma\gamma'}(nnncc,cnkk) + A_6^{\gamma\gamma'}(nnncc,nckk)\\
&\quad + A_6^{\gamma\gamma'}(nnncc,nnkk,c) + A_6^{\gamma\gamma'}(nnncc,nnkk,n) &\text{(B167)}
\end{aligned}$$

$$\begin{aligned}
A_6^{\gamma\gamma'}(cnnnn) &= A_6^{\gamma\gamma'}(cnnnn,kccc) + A_6^{\gamma\gamma'}(cnnnn,kccn) + A_6^{\gamma\gamma'}(cnnnn,kcnc)\\
&\quad + A_6^{\gamma\gamma'}(cnnnn,kncc) + A_6^{\gamma\gamma'}(cnnnn,kcnn) + A_6^{\gamma\gamma'}(cnnnn,kncn)\\
&\quad + A_6^{\gamma\gamma'}(cnnnn,knnc) + A_6^{\gamma\gamma'}(cnnnn,knnn), &\text{(B168)}
\end{aligned}$$

$$A_6^{\gamma\gamma'}(cnnnn, kcnn) = A_6^{\gamma\gamma'}(cnnnn, kcnn, kkc) + A_6^{\gamma\gamma'}(cnnnn, kcnn, kkn), \tag{B169}$$
$$A_6^{\gamma\gamma'}(cnnnn, kncn) = A_6^{\gamma\gamma'}(cnnnn, kncn, kc) + A_6^{\gamma\gamma'}(cnnnn, kncn, kn), \tag{B170}$$
$$A_6^{\gamma\gamma'}(cnnnn, knnc) = A_6^{\gamma\gamma'}(cnnnn, knnc, kck) + A_6^{\gamma\gamma'}(cnnnn, knnc, knk), \tag{B171}$$

$$\begin{aligned} A_6^{\gamma\gamma'}(cnnnn, knnn) = {} & A_6^{\gamma\gamma'}(cnnnn, knnn, kcc) + A_6^{\gamma\gamma'}(cnnnn, knnn, kcn) \\ & + A_6^{\gamma\gamma'}(cnnnn, knnn, knc) + A_6^{\gamma\gamma'}(cnnnn, knnn, knn, kc) \\ & + A_6^{\gamma\gamma'}(cnnnn, knnn, knn, kn). \end{aligned} \tag{B172}$$

$$\begin{aligned} A_6^{\gamma\gamma'}(ncnnn) = {} & A_6^{\gamma\gamma'}(ncnnn, kkcc, ckk) + A_6^{\gamma\gamma'}(ncnnn, kkcc, nkk) \\ & + A_6^{\gamma\gamma'}(ncnnn, kkcn, ckk) + A_6^{\gamma\gamma'}(ncnnn, kknc, ckk) \\ & + A_6^{\gamma\gamma'}(ncnnn, kkcn, nkk) + A_6^{\gamma\gamma'}(ncnnn, kknc, nkk) \\ & + A_6^{\gamma\gamma'}(ncnnn, kknn, ckk) + A_6^{\gamma\gamma'}(ncnnn, kknn, nkk), \end{aligned} \tag{B173}$$

$$A_6^{\gamma\gamma'}(ncnnn, kkcn, nkk) = A_6^{\gamma\gamma'}(ncnnn, kkcn, nkk, c) + A_6^{\gamma\gamma'}(ncnnn, kkcn, nkk, n), \tag{B174}$$
$$A_6^{\gamma\gamma'}(ncnnn, kknc, nkk) = A_6^{\gamma\gamma'}(ncnnn, kknc, nkk, ck) + A_6^{\gamma\gamma'}(ncnnn, kknc, nkk, nk), \tag{B175}$$
$$A_6^{\gamma\gamma'}(ncnnn, kknn, ckk) = A_6^{\gamma\gamma'}(ncnnn, kknn, ckc) + A_6^{\gamma\gamma'}(ncnnn, kknn, ckn), \tag{B176}$$

$$\begin{aligned} A_6^{\gamma\gamma'}(ncnnn, kknn, nkk) = {} & A_6^{\gamma\gamma'}(ncnnn, kknn, nkc, ck) + A_6^{\gamma\gamma'}(ncnnn, kknn, nkc, nk) \\ & + A_6^{\gamma\gamma'}(ncnnn, kknn, nkn, ck) + A_6^{\gamma\gamma'}(ncnnn, kknn, nkn, nk, c) \\ & + A_6^{\gamma\gamma'}(ncnnn, kknn, nkn, nk, n) \end{aligned} \tag{B177}$$

$$\begin{aligned} A_6^{\gamma\gamma'}(nncnn) = {} & A_6^{\gamma\gamma'}(nncnn, ckkc, kck) + A_6^{\gamma\gamma'}(nncnn, ckkc, knk) \\ & + A_6^{\gamma\gamma'}(nncnn, ckkn, kck) + A_6^{\gamma\gamma'}(nncnn, nkkc, kck) \\ & + A_6^{\gamma\gamma'}(nncnn, ckkn, knk) + A_6^{\gamma\gamma'}(nncnn, nkkc, knk) \\ & + A_6^{\gamma\gamma'}(nncnn, nkkn, kck) + A_6^{\gamma\gamma'}(nncnn, nkkn, knk), \end{aligned} \tag{B178}$$

$$A_6^{\gamma\gamma'}(nncnn, ckkn, knk) = A_6^{\gamma\gamma'}(nncnn, ckkn, knk, kc) + A_6^{\gamma\gamma'}(nncnn, ckkn, knk, kn), \tag{B179}$$
$$A_6^{\gamma\gamma'}(nncnn, nkkc, knk) = A_6^{\gamma\gamma'}(nncnn, nkkc, knk, ck) + A_6^{\gamma\gamma'}(nncnn, nkkc, knk, nk), \tag{B180}$$
$$A_6^{\gamma\gamma'}(nncnn, nkkn, kck) = A_6^{\gamma\gamma'}(nncnn, nkkn, kck, c) + A_6^{\gamma\gamma'}(nncnn, nkkn, kck, n), \tag{B181}$$

$$\begin{aligned} A_6^{\gamma\gamma'}(nncnn, nkkn, knk) = {} & A_6^{\gamma\gamma'}(nncnn, nkkn, knk, cc) + A_6^{\gamma\gamma'}(nncnn, nkkn, knk, cn) \\ & + A_6^{\gamma\gamma'}(nncnn, nkkn, knk, nc) + A_6^{\gamma\gamma'}(nncnn, nkkn, knk, nn, c) \\ & + A_6^{\gamma\gamma'}(nncnn, nkkn, knk, nn, n) \end{aligned} \tag{B182}$$

$$\begin{aligned} A_6^{\gamma\gamma'}(nnncn) = {} & A_6^{\gamma\gamma'}(nnncn, cckk, kkc) + A_6^{\gamma\gamma'}(nnncn, cckk, kkn) \\ & + A_6^{\gamma\gamma'}(nnncn, cnkk, kkc) + A_6^{\gamma\gamma'}(nnncn, nckk, kkc) \\ & + A_6^{\gamma\gamma'}(nnncn, cnkk, kkn) + A_6^{\gamma\gamma'}(nnncn, nckk, kkn) \\ & + A_6^{\gamma\gamma'}(nnncn, nnkk, kkc) + A_6^{\gamma\gamma'}(nnncn, nnkk, kkn), \end{aligned} \tag{B183}$$

$$A_6^{\gamma\gamma'}(nnncn, cnkk, kkn) = A_6^{\gamma\gamma'}(nnncn, cnkk, kkn, kc) + A_6^{\gamma\gamma'}(nnncn, cnkk, kkn, kn), \tag{B184}$$
$$A_6^{\gamma\gamma'}(nnncn, nckk, kkn) = A_6^{\gamma\gamma'}(nnncn, nckk, kkn, c) + A_6^{\gamma\gamma'}(nnncn, nckk, kkn, n), \tag{B185}$$
$$A_6^{\gamma\gamma'}(nnncn, nnkk, kkc) = A_6^{\gamma\gamma'}(nnncn, nnkk, ckc) + A_6^{\gamma\gamma'}(nnncn, nnkk, nkc), \tag{B186}$$

$$
\begin{aligned}
A_6^{\gamma\gamma'}(nnncn,nnkk,kkn) &= A_6^{\gamma\gamma'}(nnncn,nnkk,ckn,kc)+A_6^{\gamma\gamma'}(nnncn,nnkk,ckn,kn)\\
&+A_6^{\gamma\gamma'}(nnncn,nnkk,nkn,kc)+A_6^{\gamma\gamma'}(nnncn,nnkk,nkn,kn,c)\\
&+A_6^{\gamma\gamma'}(nnncn,nnkk,nkn,kn,n).
\end{aligned}
\tag{B187}
$$

$$
\begin{aligned}
A_6^{\gamma\gamma'}(nnnnc) &= A_6^{\gamma\gamma'}(nnnnc,ccck)+A_6^{\gamma\gamma'}(nnnnc,ccnk)+A_6^{\gamma\gamma'}(nnnnc,cnck)\\
&+A_6^{\gamma\gamma'}(nnnnc,ncck)+A_6^{\gamma\gamma'}(nnnnc,cnnk)+A_6^{\gamma\gamma'}(nnnnc,ncnk)\\
&+A_6^{\gamma\gamma'}(nnnnc,nnck)+A_6^{\gamma\gamma'}(nnnnc,nnnk),
\end{aligned}
\tag{B188}
$$

$$A_6^{\gamma\gamma'}(nnnnc,cnnk) = A_6^{\gamma\gamma'}(nnnnc,cnnk,kck)+A_6^{\gamma\gamma'}(nnnnc,cnnk,knk), \tag{B189}$$

$$A_6^{\gamma\gamma'}(nnnnc,ncnk) = A_6^{\gamma\gamma'}(nnnnc,ncnk,ck)+A_6^{\gamma\gamma'}(nnnnc,ncnk,nk), \tag{B190}$$

$$A_6^{\gamma\gamma'}(nnnnc,nnck) = A_6^{\gamma\gamma'}(nnnnc,nnck,c)+A_6^{\gamma\gamma'}(nnnnc,nnck,n), \tag{B191}$$

$$
\begin{aligned}
A_6^{\gamma\gamma'}(nnnnc,nnnk) &= A_6^{\gamma\gamma'}(nnnnc,nnnk,cck)+A_6^{\gamma\gamma'}(nnnnc,nnnk,cnk)\\
&+A_6^{\gamma\gamma'}(nnnnc,nnnk,nck)+A_6^{\gamma\gamma'}(nnnnc,nnnk,nnk,ck)\\
&+A_6^{\gamma\gamma'}(nnnnc,nnnk,nnk,nk).
\end{aligned}
\tag{B192}
$$

$$
\begin{aligned}
A_6^{\gamma\gamma'}(nnnnn) &= A_6^{\gamma\gamma'}(nnnnn,cccc)+A_6^{\gamma\gamma'}(nnnnn,cccn)+A_6^{\gamma\gamma'}(nnnnn,ccnc)\\
&+A_6^{\gamma\gamma'}(nnnnn,cncc)+A_6^{\gamma\gamma'}(nnnnn,nccc)+A_6^{\gamma\gamma'}(nnnnn,ccnn)\\
&+A_6^{\gamma\gamma'}(nnnnn,cncn)+A_6^{\gamma\gamma'}(nnnnn,cnnc)+A_6^{\gamma\gamma'}(nnnnn,nccn)\\
&+A_6^{\gamma\gamma'}(nnnnn,ncnc)+A_6^{\gamma\gamma'}(nnnnn,nncc)+A_6^{\gamma\gamma'}(nnnnn,cnnn)\\
&+A_6^{\gamma\gamma'}(nnnnn,ncnn)+A_6^{\gamma\gamma'}(nnnnn,nncn)+A_6^{\gamma\gamma'}(nnnnn,nnnc)\\
&+A_6^{\gamma\gamma'}(nnnnn,nnnn),
\end{aligned}
\tag{B193}
$$

$$A_6^{\gamma\gamma'}(nnnnn,ccnn) = A_6^{\gamma\gamma'}(nnnnn,ccnn,kkc)+A_6^{\gamma\gamma'}(nnnnn,ccnn,kkn), \tag{B194}$$

$$A_6^{\gamma\gamma'}(nnnnn,cncn) = A_6^{\gamma\gamma'}(nnnnn,cncn,kc)+A_6^{\gamma\gamma'}(nnnnn,cncn,kn), \tag{B195}$$

$$A_6^{\gamma\gamma'}(nnnnn,cnnc) = A_6^{\gamma\gamma'}(nnnnn,cnnc,kck)+A_6^{\gamma\gamma'}(nnnnn,cnnc,knk), \tag{B196}$$

$$A_6^{\gamma\gamma'}(nnnnn,nccn) = A_6^{\gamma\gamma'}(nnnnn,nccn,c)+A_6^{\gamma\gamma'}(nnnnn,nccn,n), \tag{B197}$$

$$A_6^{\gamma\gamma'}(nnnnn,ncnc) = A_6^{\gamma\gamma'}(nnnnn,ncnc,ck)+A_6^{\gamma\gamma'}(nnnnn,ncnc,nk), \tag{B198}$$

$$A_6^{\gamma\gamma'}(nnnnn,nncc) = A_6^{\gamma\gamma'}(nnnnn,nncc,ckk)+A_6^{\gamma\gamma'}(nnnnn,nncc,nkk), \tag{B199}$$

$$
\begin{aligned}
A_6^{\gamma\gamma'}(nnnnn,cnnn) &= A_6^{\gamma\gamma'}(nnnnn,cnnn,kcc)+A_6^{\gamma\gamma'}(nnnnn,cnnn,kcn)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,cnnn,knc)+A_6^{\gamma\gamma'}(nnnnn,cnnn,knn,kc)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,cnnn,knn,kn), \qquad (B200)\\
A_6^{\gamma\gamma'}(nnnnn,ncnn) &= A_6^{\gamma\gamma'}(nnnnn,ncnn,kkc,ck)+A_6^{\gamma\gamma'}(nnnnn,ncnn,kkc,nk)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,ncnn,kkn,ck)+A_6^{\gamma\gamma'}(nnnnn,ncnn,kkn,nk,c)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,ncnn,kkn,nk,n), \qquad (B201)\\
A_6^{\gamma\gamma'}(nnnnn,nncn) &= A_6^{\gamma\gamma'}(nnnnn,nncn,ckk,kc)+A_6^{\gamma\gamma'}(nnnnn,nncn,ckk,kn)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,nncn,nkk,kc)+A_6^{\gamma\gamma'}(nnnnn,nncn,nkk,kn,c)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,nncn,nkk,kn,n), \qquad (B202)\\
A_6^{\gamma\gamma'}(nnnnn,nnnc) &= A_6^{\gamma\gamma'}(nnnnn,nnnc,cck)+A_6^{\gamma\gamma'}(nnnnn,nnnc,cnk)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,nnnc,nck)+A_6^{\gamma\gamma'}(nnnnn,nnnc,nnk,ck)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,nnnc,nnk,nk), \qquad (B203)\\
A_6^{\gamma\gamma'}(nnnnn,nnnn) &= A_6^{\gamma\gamma'}(nnnnn,nnnn,ccc)+A_6^{\gamma\gamma'}(nnnnn,nnnn,ccn)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,nnnn,cnc)+A_6^{\gamma\gamma'}(nnnnn,nnnn,ncc)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,nnnn,cnn)+A_6^{\gamma\gamma'}(nnnnn,nnnn,ncn)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,nnnn,nnc)+A_6^{\gamma\gamma'}(nnnnn,nnnn,nnn), \qquad (B204)
\end{aligned}
$$

$$A_6^{\gamma\gamma'}(nnnnn,nnnn,cnn)=A_6^{\gamma\gamma'}(nnnnn,nnnn,cnn,kc)+A_6^{\gamma\gamma'}(nnnnn,nnnn,cnn,kn), \qquad (B205)$$
$$A_6^{\gamma\gamma'}(nnnnn,nnnn,ncn)=A_6^{\gamma\gamma'}(nnnnn,nnnn,ncn,c)+A_6^{\gamma\gamma'}(nnnnn,nnnn,ncn,n), \qquad (B206)$$
$$A_6^{\gamma\gamma'}(nnnnn,nnnn,nnc)=A_6^{\gamma\gamma'}(nnnnn,nnnn,nnc,ck)+A_6^{\gamma\gamma'}(nnnnn,nnnn,nnc,nk), \qquad (B207)$$

$$
\begin{aligned}
A_6^{\gamma\gamma'}(nnnnn,nnnn,nnn) &= A_6^{\gamma\gamma'}(nnnnn,nnnn,nnn,cc)+A_6^{\gamma\gamma'}(nnnnn,nnnn,nnn,cn)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,nnnn,nnn,nc)+A_6^{\gamma\gamma'}(nnnnn,nnnn,nnn,nn,c)\\
&\quad+A_6^{\gamma\gamma'}(nnnnn,nnnn,nnn,nn,n). \qquad (B208)
\end{aligned}
$$

Thus, we obtain the contribution from the six order approximation made of 203 terms after finding out all of contractions and anti-contractions.

Just like we have done in the $l=4$ or 5 cases, we decompose

$$
\begin{aligned}
A_6^{\gamma\gamma'} &= A_6^{\gamma\gamma'}(\mathrm{e})+A_6^{\gamma\gamma'}(t\mathrm{e})+A_6^{\gamma\gamma'}(t^2\mathrm{e})+A_6^{\gamma\gamma'}(t^3\mathrm{e}) \qquad (B209)\\
&= A_6^{\gamma\gamma'}(\mathrm{e},t\mathrm{e})+A_6^{\gamma\gamma'}(t^2\mathrm{e},t^3\mathrm{e}), \qquad (B210)
\end{aligned}
$$

To our purpose, we only calculate the second term $A_6^{\gamma\gamma'}(t^2\mathrm{e},t^3\mathrm{e})$ in eq.(B210). Without loss of generality, we decompose it into

$$
\begin{aligned}
A_6^{\gamma\gamma'}(t^2\mathrm{e},t^3\mathrm{e}) &= A_6^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},t^3\mathrm{e}^{-\mathrm{i}E_\gamma t})+A_6^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t},t^3\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t})\\
&\quad+A_6^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'} t},t^3\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}). \qquad (B211)
\end{aligned}
$$

While, every term in above equations has its diagonal and non-diagonal parts about γ and γ', that is

$$
\begin{aligned}
A_6^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t}) &= A_6^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{D})+A_6^{\gamma\gamma'}(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{N}), \qquad (B212)\\
A_6^{\gamma\gamma'}(t^3\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t}) &= A_6^{\gamma\gamma'}(t^3\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{D})+A_6^{\gamma\gamma'}(t^3\mathrm{e}^{-\mathrm{i}E_{\gamma_i} t};\mathrm{N}). \qquad (B213)
\end{aligned}
$$

where E_{γ_i} takes E_γ, E_{γ_1} and $E_{\gamma'}$. In the following, we respectively calculate them term by term, and we only write down the non-zero expressions for saving space.

$$A_6(ccccc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1}\left[\frac{(-\mathrm{i}t)^3}{3!}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^3}-\frac{(-\mathrm{i}t)^2}{2}\frac{3\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^4}\right.$$
$$\left.+\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^4}\right]|g_1^{\gamma\gamma_1}|^6\,\delta_{\gamma\gamma'}. \tag{B214}$$

$$A_6(ccccn;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1}\left[\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^3\,(E_\gamma-E_{\gamma'})}\right.$$
$$\left.-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^3\,(E_{\gamma_1}-E_{\gamma'})}\right]|g_1^{\gamma\gamma_1}|^4\,g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}. \tag{B215}$$

$$A_6(cccnc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2}\left[\frac{(-\mathrm{i}t)^3}{3!}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2\,(E_\gamma-E_{\gamma'})}-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2\,(E_\gamma-E_{\gamma_2})^2}\right.$$
$$\left.-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^3\,(E_\gamma-E_{\gamma_2})}\right]|g_1^{\gamma\gamma_1}|^4\,|g_1^{\gamma\gamma_2}|^2\,\eta_{\gamma_1\gamma_2}\delta_{\gamma\gamma'}. \tag{B216}$$

$$A_6(ccncc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^2\,(E_{\gamma_1}-E_{\gamma'})^2}\,|g_1^{\gamma\gamma_1}|^2\left|g_1^{\gamma_1\gamma'}\right|^2g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}. \tag{B217}$$

$$A_6(cnccc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2}\left[\frac{(-\mathrm{i}t)^3}{3!}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})\,(E_\gamma-E_{\gamma_2})^2}-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})\,(E_\gamma-E_{\gamma_2})^3}\right.$$
$$\left.-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2\,(E_\gamma-E_{\gamma_2})^2}\right]|g_1^{\gamma\gamma_1}|^2\,|g_1^{\gamma\gamma_2}|^4\,\eta_{\gamma_1\gamma_2}\delta_{\gamma\gamma'}. \tag{B218}$$

$$A_6(ncccc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1}\left[-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})\,(E_{\gamma_1}-E_{\gamma'})^3}\right.$$
$$\left.+\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})\,(E_{\gamma_1}-E_{\gamma'})^3}\right]\left|g_1^{\gamma_1\gamma'}\right|^4g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}. \tag{B219}$$

$$A_6(cccnn,kkkc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1}\left[\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})\,(E_\gamma-E_{\gamma'})^3}\right.$$
$$\left.+\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{(E_\gamma-E_{\gamma'})^3\,(E_{\gamma_1}-E_{\gamma'})}\right]\left|g_1^{\gamma\gamma'}\right|^4g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}. \tag{B220}$$

$$A_6(cccnn,kkkn;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}\,|g_1^{\gamma\gamma_1}|^4\,g_1^{\gamma\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma'}}{(E_\gamma-E_{\gamma_1})^2\,(E_\gamma-E_{\gamma_2})\,(E_\gamma-E_{\gamma'})}. \tag{B221}$$

$$A_6(ccncn,kkc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2}\left[\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^3\,(E_\gamma-E_{\gamma'})}\right.$$
$$\left.-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{(E_\gamma-E_{\gamma_1})^3\,(E_{\gamma_1}-E_{\gamma_2})}\right]|g_1^{\gamma\gamma_1}|^4\,|g_1^{\gamma_1\gamma_2}|^2\,\eta_{\gamma\gamma_2}\delta_{\gamma\gamma'}. \tag{B222}$$

$$A_6(ccncn,kkn;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}\left|g_1^{\gamma\gamma_1}\right|^2\left|g_1^{\gamma_1\gamma_2}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}\eta_{\gamma_2\gamma'}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_{\gamma_1}-E_{\gamma_2}\right)\left(E_{\gamma_1}-E_{\gamma'}\right)}. \tag{B223}$$

$$A_6(ccnnc,kkck;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}\left|g_1^{\gamma\gamma_1}\right|^2\left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_\gamma-E_{\gamma'}\right)^2}. \tag{B224}$$

$$A_6(cnccn,kc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}\left|g_1^{\gamma\gamma_1}\right|^2\left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_\gamma-E_{\gamma'}\right)^2}. \tag{B225}$$

$$A_6(cnccn,kn;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}\left|g_1^{\gamma\gamma_1}\right|^2\left|g_1^{\gamma\gamma_2}\right|^2 g_1^{\gamma\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma'}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma_2}\right)^2\left(E_\gamma-E_{\gamma'}\right)}. \tag{B226}$$

$$\begin{aligned}A_6(cncnc,kck;t^2\mathrm{e},t^3\mathrm{e}) \;=\; &\sum_{\gamma_1,\gamma_2}\Bigg[\frac{(-\mathrm{i}t)^3}{3!}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_\gamma-E_{\gamma_2}\right)}\\&-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_\gamma-E_{\gamma_2}\right)^2}\\&-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_1}\right)^3\left(E_\gamma-E_{\gamma_2}\right)}\Bigg]\\&\times\left|g_1^{\gamma\gamma_1}\right|^4\left|g_1^{\gamma\gamma_2}\right|^2\eta_{\gamma_1\gamma_2}\delta_{\gamma\gamma'}.\end{aligned} \tag{B227}$$

$$\begin{aligned}A_6(cncnc,knk;t^2\mathrm{e},t^3\mathrm{e}) \;=\; &\sum_{\gamma_1,\gamma_2,\gamma_3}\Bigg[\frac{(-\mathrm{i}t)^3}{3!}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma_2}\right)\left(E_\gamma-E_{\gamma_3}\right)}\\&-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma_2}\right)\left(E_\gamma-E_{\gamma_3}\right)^2}\\&-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_\gamma-E_{\gamma_2}\right)^2\left(E_\gamma-E_{\gamma_3}\right)}\\&-\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_\gamma-E_{\gamma_2}\right)\left(E_\gamma-E_{\gamma_3}\right)}\Bigg]\\&\times\left|g_1^{\gamma\gamma_1}\right|^2\left|g_1^{\gamma\gamma_2}\right|^2\left|g_1^{\gamma\gamma_3}\right|^2\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_3}\delta_{\gamma\gamma'}.\end{aligned} \tag{B228}$$

$$A_6(cnncc,kckk;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}\left|g_1^{\gamma_1\gamma'}\right|^2\left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}}{\left(E_\gamma-E_{\gamma'}\right)^2\left(E_{\gamma_1}-E_{\gamma'}\right)^2}. \tag{B229}$$

$$A_6(ncccn,c;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}\left|g_1^{\gamma\gamma_1}\right|^2\left|g_1^{\gamma_1\gamma_2}\right|^4\eta_{\gamma\gamma_2}\delta_{\gamma\gamma'}}{\left(E_\gamma-E_{\gamma_1}\right)^2\left(E_{\gamma_1}-E_{\gamma_2}\right)^2}. \tag{B230}$$

$$A_6(ncccn,n;t^2\mathrm{e},t^3\mathrm{e}) \;=\; -\sum_{\gamma_1,\gamma_2}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}\left|g_1^{\gamma_1\gamma_2}\right|^4 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma_2\gamma'}}{\left(E_\gamma-E_{\gamma_1}\right)\left(E_{\gamma_1}-E_{\gamma_2}\right)^2\left(E_{\gamma_1}-E_{\gamma'}\right)}. \tag{B231}$$

$$A_6(nccnc,ck;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}\left|g_1^{\gamma\gamma'}\right|^2\left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}}{\left(E_\gamma-E_{\gamma'}\right)^2\left(E_{\gamma_1}-E_{\gamma'}\right)^2}. \tag{B232}$$

$$A_6(nccnc, nk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma_1\gamma'}\right|^2 \left|g_1^{\gamma_2\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_2}}{\left(E_\gamma - E_{\gamma'}\right)\left(E_{\gamma_1} - E_{\gamma'}\right)^2\left(E_{\gamma_2} - E_{\gamma'}\right)}. \tag{B233}$$

$$\begin{aligned} A_6(ncncc, ckk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \Bigg[& \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma - E_{\gamma_1}\right)^3\left(E_\gamma - E_{\gamma_2}\right)} \\ & - \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}}{\left(E_\gamma - E_{\gamma_1}\right)^3\left(E_{\gamma_1} - E_{\gamma_2}\right)} \Bigg] |g_1^{\gamma\gamma_1}|^4 \, |g_1^{\gamma_1\gamma_2}|^2 \, \eta_{\gamma\gamma_2}\delta_{\gamma\gamma'}. \end{aligned} \tag{B234}$$

$$A_6(ncncc, nkk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; -\sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t} \, |g_1^{\gamma_1\gamma_2}|^2 \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma_2\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)\left(E_{\gamma_1} - E_{\gamma_2}\right)\left(E_{\gamma_1} - E_{\gamma'}\right)^2}. \tag{B235}$$

$$\begin{aligned} A_6(nnccc, ckkk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \Bigg[& \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma - E_{\gamma_1}\right)\left(E_\gamma - E_{\gamma'}\right)^3} \\ & + \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{\left(E_\gamma - E_{\gamma'}\right)^3\left(E_{\gamma_1} - E_{\gamma'}\right)} \Bigg] \left|g_1^{\gamma\gamma'}\right|^4 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}. \end{aligned} \tag{B236}$$

$$A_6(nnccc, nkkk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma_2\gamma'}\right|^4 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_2}}{\left(E_\gamma - E_{\gamma'}\right)\left(E_{\gamma_1} - E_{\gamma'}\right)\left(E_{\gamma_2} - E_{\gamma'}\right)^2}. \tag{B237}$$

$$\begin{aligned} A_6(ccnnn, kkcc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \Bigg[& \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{\left(E_\gamma - E_{\gamma_1}\right)\left(E_\gamma - E_{\gamma'}\right)^3} \\ & + \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}}{\left(E_\gamma - E_{\gamma'}\right)^3\left(E_{\gamma_1} - E_{\gamma'}\right)} \Bigg] \left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}. \end{aligned} \tag{B238}$$

$$A_6(ccnnn, kkcn; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \, |g_1^{\gamma\gamma_1}|^2 \, g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)^2\left(E_\gamma - E_{\gamma_2}\right)\left(E_\gamma - E_{\gamma'}\right)}. \tag{B239}$$

$$A_6(ccnnn, kknc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}}{\left(E_\gamma - E_{\gamma'}\right)^2\left(E_{\gamma_1} - E_{\gamma'}\right)\left(E_{\gamma_2} - E_{\gamma'}\right)}. \tag{B240}$$

$$A_6(ccnnn, kknn, kkc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \, |g_1^{\gamma\gamma_1}|^2 \, g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_3}\delta_{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)^2\left(E_\gamma - E_{\gamma_2}\right)\left(E_\gamma - E_{\gamma_3}\right)}. \tag{B241}$$

$$A_6(cncnn, kkkc, kck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \, |g_1^{\gamma\gamma_1}|^2 \left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)^2\left(E_\gamma - E_{\gamma'}\right)^2}. \tag{B242}$$

$$A_6(cncnn, kkkc, knk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \, |g_1^{\gamma\gamma_1}|^2 \left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)\left(E_\gamma - E_{\gamma_2}\right)\left(E_\gamma - E_{\gamma'}\right)^2}. \tag{B243}$$

$$A_6(cncnn, kkkn, kck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}\,|g_1^{\gamma\gamma_1}|^2\,|g_1^{\gamma\gamma_2}|^2\,g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma_2\gamma'}}{(E_\gamma-E_{\gamma_1})^2\,(E_\gamma-E_{\gamma_2})\,(E_\gamma-E_{\gamma'})}. \tag{B244}$$

$$A_6(cncnn, kkkn, knk, kc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}\,|g_1^{\gamma\gamma_1}|^2\,\left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma'}}{(E_\gamma-E_{\gamma_1})\,(E_\gamma-E_{\gamma_2})\,(E_\gamma-E_{\gamma'})^2}. \tag{B245}$$

$$\begin{aligned} &A_6(cncnn, kkkn, knk, kn; t^2\mathrm{e}, t^3\mathrm{e}) \\ &= \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}\,|g_1^{\gamma\gamma_1}|^2\,|g_1^{\gamma\gamma_2}|^2\,g_1^{\gamma\gamma_3}g_1^{\gamma_3\gamma'}\eta_{\gamma\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma_3}\eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma_3}\eta_{\gamma_2\gamma'}}{(E_\gamma-E_{\gamma_1})\,(E_\gamma-E_{\gamma_2})\,(E_\gamma-E_{\gamma_3})\,(E_\gamma-E_{\gamma'})}. \end{aligned} \tag{B246}$$

$$A_6(cnncn, kckk, kkc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}\,|g_1^{\gamma\gamma_1}|^2\,|g_1^{\gamma_1\gamma_2}|^2\,|g_1^{\gamma\gamma_2}|^2\,\delta_{\gamma\gamma'}}{(E_\gamma-E_{\gamma_1})^2\,(E_\gamma-E_{\gamma_2})^2} \tag{B247}$$

$$A_6(cnncn, knkk, kkc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}\,|g_1^{\gamma\gamma_1}|^2\,|g_1^{\gamma\gamma_2}|^2\,|g_1^{\gamma_2\gamma_3}|^2\,\eta_{\gamma\gamma_3}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma_3}\delta_{\gamma\gamma'}}{(E_\gamma-E_{\gamma_1})\,(E_\gamma-E_{\gamma_2})^2\,(E_\gamma-E_{\gamma_3})}. \tag{B248}$$

$$\begin{aligned} A_6(cnnnc, kcck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \Bigg[&\frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})\,(E_\gamma-E_{\gamma'})^3} \\ &+\frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})^3\,(E_{\gamma_1}-E_{\gamma'})} \Bigg] \left|g_1^{\gamma\gamma'}\right|^4 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}. \end{aligned} \tag{B249}$$

$$A_6(cnnnc, kcnk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}\left|g_1^{\gamma\gamma'}\right|^2 \left|g_1^{\gamma_2\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_2}}{(E_\gamma-E_{\gamma'})^2\,(E_{\gamma_1}-E_{\gamma'})\,(E_{\gamma_2}-E_{\gamma'})}. \tag{B250}$$

$$A_6(cnnnc, knck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}\,|g_1^{\gamma\gamma_1}|^2 \left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma'}}{(E_\gamma-E_{\gamma_1})\,(E_\gamma-E_{\gamma_2})\,(E_{\gamma_2}-E_{\gamma'})^2}. \tag{B251}$$

$$A_6(nccnn, kkkc, ck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}\left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'}g_1^{\gamma'\gamma_1}g_1^{\gamma_1\gamma}g_1^{\gamma\gamma'}}{(E_\gamma-E_{\gamma'})^2\,(E_{\gamma_1}-E_{\gamma'})^2}. \tag{B252}$$

$$A_6(nccnn, kkkc, nk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}\left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'}g_1^{\gamma'\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma'}\eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}}{(E_\gamma-E_{\gamma'})\,(E_{\gamma_1}-E_{\gamma'})^2\,(E_{\gamma_2}-E_{\gamma'})}. \tag{B253}$$

$$A_6(ncncn, ckc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}\,|g_1^{\gamma\gamma_1}|^2 \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}}{(E_\gamma-E_{\gamma_1})^2\,(E_{\gamma_1}-E_{\gamma'})^2}. \tag{B254}$$

$$A_6(ncncn, ckn; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}\,|g_1^{\gamma\gamma_1}|^2\,|g_1^{\gamma_1\gamma_2}|^2\,g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}\eta_{\gamma_2\gamma'}}{(E_\gamma-E_{\gamma_1})^2\,(E_{\gamma_1}-E_{\gamma_2})\,(E_{\gamma_1}-E_{\gamma'})}. \tag{B255}$$

$$A_6(ncncn, nkc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; -\sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}\left|g_1^{\gamma_1\gamma'}\right|^2 |g_1^{\gamma_1\gamma_2}|^2\, g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma'}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma'}\eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1})\,(E_{\gamma_1} - E_{\gamma_2})\,(E_{\gamma_1} - E_{\gamma'})^2}. \tag{B256}$$

$$A_6(ncncn, nkn, c; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}\,|g_1^{\gamma\gamma_1}|^2\,|g_1^{\gamma_1\gamma_2}|^2\,|g_1^{\gamma_1\gamma_3}|^2\,\eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma_2\gamma_3}\delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2\,(E_{\gamma_1} - E_{\gamma_2})\,(E_{\gamma_1} - E_{\gamma_3})}. \tag{B257}$$

$$\begin{aligned} &A_6(ncncn, nkn, n; t^2\mathrm{e}, t^3\mathrm{e}) \\ &\quad = -\sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma_1}t}\,|g_1^{\gamma_1\gamma_2}|^2\,|g_1^{\gamma_1\gamma_3}|^2\, g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma\gamma'}\eta_{\gamma_2\gamma_3}\eta_{\gamma_2\gamma'}\eta_{\gamma_3\gamma'}}{(E_\gamma - E_{\gamma_1})\,(E_{\gamma_1} - E_{\gamma_2})\,(E_{\gamma_1} - E_{\gamma_3})\,(E_{\gamma_1} - E_{\gamma'})}. \end{aligned} \tag{B258}$$

$$A_6(ncnnc, kkck, ckk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}\,|g_1^{\gamma\gamma_1}|^2\,|g_1^{\gamma\gamma_2}|^2\,|g_1^{\gamma_1\gamma_2}|^2\,\delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2\,(E_\gamma - E_{\gamma_2})^2}. \tag{B259}$$

$$A_6(ncnnc, kknk, ckk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}\,|g_1^{\gamma\gamma_1}|^2\,|g_1^{\gamma\gamma_3}|^2\,|g_1^{\gamma_1\gamma_2}|^2\,\eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_3}\delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2\,(E_\gamma - E_{\gamma_2})\,(E_\gamma - E_{\gamma_3})}. \tag{B260}$$

$$A_6(nnccn, cc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}\,|g_1^{\gamma\gamma_1}|^2\, g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2\,(E_\gamma - E_{\gamma'})^2}. \tag{B261}$$

$$A_6(nnccn, cn; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma}t}\,|g_1^{\gamma\gamma_2}|^2\, g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1})\,(E_\gamma - E_{\gamma_2})^2\,(E_\gamma - E_{\gamma'})}. \tag{B262}$$

$$A_6(nncnc, ckkk, kck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}\left|g_1^{\gamma\gamma'}\right|^2 \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{(E_\gamma - E_{\gamma'})^2\,(E_{\gamma_1} - E_{\gamma'})^2}. \tag{B263}$$

$$A_6(nncnc, ckkk, knk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}\left|g_1^{\gamma\gamma'}\right|^2 \left|g_1^{\gamma_2\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_2}}{(E_\gamma - E_{\gamma'})^2\,(E_{\gamma_1} - E_{\gamma'})\,(E_{\gamma_2} - E_{\gamma'})}. \tag{B264}$$

$$A_6(nncnc, nkkk, kck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}\left|g_1^{\gamma_1\gamma'}\right|^2 \left|g_1^{\gamma_2\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_2}\eta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma'})\,(E_{\gamma_1} - E_{\gamma'})^2\,(E_{\gamma_2} - E_{\gamma'})}. \tag{B265}$$

$$A_6(nncnc, nkkk, knk, ck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}\left|g_1^{\gamma\gamma'}\right|^2 \left|g_1^{\gamma_2\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_2}}{(E_\gamma - E_{\gamma'})^2\,(E_{\gamma_1} - E_{\gamma'})\,(E_{\gamma_2} - E_{\gamma'})}. \tag{B266}$$

$$\begin{aligned} &A_6(nncnc, nkkk, knk, nk; t^2\mathrm{e}, t^3\mathrm{e}) \\ &\quad = \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t}\left|g_1^{\gamma_2\gamma'}\right|^2 \left|g_1^{\gamma_3\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma\gamma'}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_3}}{(E_\gamma - E_{\gamma'})\,(E_{\gamma_1} - E_{\gamma'})\,(E_{\gamma_2} - E_{\gamma'})\,(E_{\gamma_3} - E_{\gamma'})}. \end{aligned} \tag{B267}$$

$$A_6(nnncc, cckk; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1} \left[\frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma'})^3} + \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma - E_{\gamma'})^3 (E_{\gamma_1} - E_{\gamma'})} \right] \left| g_1^{\gamma\gamma'} \right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}. \tag{B268}$$

$$A_6(nnncc, cnkk; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left| g_1^{\gamma\gamma'} \right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2}. \tag{B269}$$

$$A_6(nnncc, nckk; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t} \left| g_1^{\gamma_2\gamma'} \right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_1} \eta_{\gamma\gamma_2}}{(E_\gamma - E_{\gamma'})(E_{\gamma_1} - E_{\gamma'})(E_{\gamma_2} - E_{\gamma'})^2}. \tag{B270}$$

$$A_6(nnncc, nnkk, c; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} |g_1^{\gamma\gamma_3}|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma\gamma_2} \eta_{\gamma_1\gamma_3} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})^2}. \tag{B271}$$

$$A_6(cnnnn, kccc; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma'})^2}. \tag{B272}$$

$$A_6(cnnnn, kccn; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})}. \tag{B273}$$

$$A_6(cnnnn, kncc; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_1\gamma_2}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2}. \tag{B274}$$

$$A_6(cnnnn, kcnn, kkc; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_1} g_1^{\gamma_1\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_2\gamma_3} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})}. \tag{B275}$$

$$A_6(cnnnn, kncn, kc; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left| g_1^{\gamma\gamma'} \right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma'})^2}. \tag{B276}$$

$$A_6(cnnnn, kncn, kn; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma'} \eta_{\gamma_2\gamma'} \eta_{\gamma_3\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma'})}. \tag{B277}$$

$$A_6(cnnnn, knnn, kcc; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_1} g_1^{\gamma_1\gamma} \eta_{\gamma\gamma_3} \eta_{\gamma_1\gamma_2} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})^2 (E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})}. \tag{B278}$$

$$A_6(cnnnn, knnn, knc; t^2\mathrm{e}, t^3\mathrm{e}) = \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} |g_1^{\gamma\gamma_1}|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma_4} g_1^{\gamma_4\gamma} \eta_{\gamma\gamma_3} \eta_{\gamma_1\gamma_2} \eta_{\gamma_1\gamma_3} \eta_{\gamma_1\gamma_4} \eta_{\gamma_2\gamma_4} \delta_{\gamma\gamma'}}{(E_\gamma - E_{\gamma_1})(E_\gamma - E_{\gamma_2})(E_\gamma - E_{\gamma_3})(E_\gamma - E_{\gamma_4})}. \tag{B279}$$

$$A_6(ncnnn,kkcc,ckk;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma\gamma'}\right|^2 \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{\left(E_\gamma - E_{\gamma'}\right)^2 \left(E_{\gamma_1} - E_{\gamma'}\right)^2}. \tag{B280}$$

$$A_6(ncnnn,kkcc,nkk;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}}{\left(E_\gamma - E_{\gamma'}\right) \left(E_{\gamma_1} - E_{\gamma'}\right)^2 \left(E_{\gamma_2} - E_{\gamma'}\right)}. \tag{B281}$$

$$A_6(ncnnn,kknc,ckk;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma\gamma'}\right|^2 \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_1}\eta_{\gamma_1\gamma_2}}{\left(E_\gamma - E_{\gamma'}\right)^2 \left(E_{\gamma_1} - E_{\gamma'}\right) \left(E_{\gamma_2} - E_{\gamma'}\right)}. \tag{B282}$$

$$A_6(ncnnn,kknc,nkk,ck;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma\gamma_1}\eta_{\gamma_1\gamma_2}}{\left(E_\gamma - E_{\gamma'}\right)^2 \left(E_{\gamma_1} - E_{\gamma'}\right) \left(E_{\gamma_2} - E_{\gamma'}\right)}. \tag{B283}$$

$$\begin{aligned} &A_6(ncnnn,kknc,nkk,nk;t^2\mathrm{e},t^3\mathrm{e}) \\ &= \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma'} \eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma_1\gamma_2}\eta_{\gamma_1\gamma_3}}{\left(E_\gamma - E_{\gamma'}\right) \left(E_{\gamma_1} - E_{\gamma'}\right) \left(E_{\gamma_2} - E_{\gamma'}\right) \left(E_{\gamma_3} - E_{\gamma'}\right)}. \end{aligned} \tag{B284}$$

$$A_6(nncnn,ckkc,kck;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left|g_1^{\gamma\gamma_1}\right|^2 \left|g_1^{\gamma\gamma_2}\right|^2 \left|g_1^{\gamma_1\gamma_2}\right|^2 \delta_{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)^2 \left(E_\gamma - E_{\gamma_2}\right)^2}. \tag{B285}$$

$$A_6(nncnn,ckkc,knk;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left|g_1^{\gamma\gamma_2}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_1\gamma_3}\delta_{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right) \left(E_\gamma - E_{\gamma_2}\right)^2 \left(E_\gamma - E_{\gamma_3}\right)}. \tag{B286}$$

$$A_6(nnncn,cckk,kkc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left|g_1^{\gamma\gamma_1}\right|^2 \left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)^2 \left(E_\gamma - E_{\gamma'}\right)^2}. \tag{B287}$$

$$A_6(nnncn,cckk,kkn;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left|g_1^{\gamma\gamma_1}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)^2 \left(E_\gamma - E_{\gamma_2}\right) \left(E_\gamma - E_{\gamma'}\right)}. \tag{B288}$$

$$A_6(nnncn,cnkk,kkc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left|g_1^{\gamma\gamma_2}\right|^2 \left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma'} \eta_{\gamma_1\gamma_2}\eta_{\gamma_2\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right) \left(E_\gamma - E_{\gamma_2}\right) \left(E_\gamma - E_{\gamma'}\right)^2}. \tag{B289}$$

$$A_6(nnncn,cnkk,kkn,kc;t^2\mathrm{e},t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left|g_1^{\gamma\gamma_2}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_2}\eta_{\gamma_2\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right) \left(E_\gamma - E_{\gamma_2}\right) \left(E_\gamma - E_{\gamma'}\right)^2}. \tag{B290}$$

$$\begin{aligned} &A_6(nnncn,cnkk,kkn,kn;t^2\mathrm{e},t^3\mathrm{e}) \\ &= \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left|g_1^{\gamma\gamma_3}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma'} \eta_{\gamma_1\gamma_3}\eta_{\gamma_1\gamma'}\eta_{\gamma_2\gamma_3}\eta_{\gamma_2\gamma'}\eta_{\gamma_3\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right) \left(E_\gamma - E_{\gamma_2}\right) \left(E_\gamma - E_{\gamma_3}\right) \left(E_\gamma - E_{\gamma'}\right)}. \end{aligned} \tag{B291}$$

$$A_6(nnnnc, ccck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma'}\right)^2 \left(E_{\gamma_1} - E_{\gamma'}\right)^2}. \tag{B292}$$

$$A_6(nnnnc, ccnk; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma_2\gamma'}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma} g_1^{\gamma\gamma'} \eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_2}}{\left(E_\gamma - E_{\gamma'}\right)^2 \left(E_{\gamma_1} - E_{\gamma'}\right) \left(E_{\gamma_2} - E_{\gamma'}\right)}. \tag{B293}$$

$$A_6(nnnnc, ncck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma_1\gamma'}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}}{\left(E_\gamma - E_{\gamma'}\right) \left(E_{\gamma_1} - E_{\gamma'}\right)^2 \left(E_{\gamma_2} - E_{\gamma'}\right)}. \tag{B294}$$

$$A_6(nnnnc, ncnk, ck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma\gamma'}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}}{\left(E_\gamma - E_{\gamma'}\right)^2 \left(E_{\gamma_1} - E_{\gamma'}\right) \left(E_{\gamma_2} - E_{\gamma'}\right)}. \tag{B295}$$

$$\begin{aligned} &A_6(nnnnc, ncnk, nk; t^2\mathrm{e}, t^3\mathrm{e}) \\ &= \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'}t} \left|g_1^{\gamma_3\gamma'}\right|^2 g_1^{\gamma\gamma'} g_1^{\gamma'\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma'} \eta_{\gamma\gamma_1}\eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma_1\gamma_3}\eta_{\gamma_2\gamma_3}}{\left(E_\gamma - E_{\gamma'}\right) \left(E_{\gamma_1} - E_{\gamma'}\right) \left(E_{\gamma_2} - E_{\gamma'}\right) \left(E_{\gamma_3} - E_{\gamma'}\right)}. \end{aligned} \tag{B296}$$

$$A_6(nnnnc, nnck, c; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left|g_1^{\gamma\gamma_2}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_1\gamma_3}\delta_{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right) \left(E_\gamma - E_{\gamma_2}\right)^2 \left(E_\gamma - E_{\gamma_3}\right)}. \tag{B297}$$

$$A_6(nnnnc, nnnk, cck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left|g_1^{\gamma\gamma_1}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma\gamma_2}\eta_{\gamma_1\gamma_3}\delta_{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)^2 \left(E_\gamma - E_{\gamma_2}\right) \left(E_\gamma - E_{\gamma_3}\right)}. \tag{B298}$$

$$\begin{aligned} &A_6(nnnnc, nnnk, cnk; t^2\mathrm{e}, t^3\mathrm{e}) \\ &= \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left|g_1^{\gamma\gamma_4}\right|^2 g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma\gamma_2}\eta_{\gamma\gamma_3}\eta_{\gamma_1\gamma_3}\eta_{\gamma_1\gamma_4}\eta_{\gamma_2\gamma_4}\eta_{\gamma_3\gamma_4}\delta_{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right) \left(E_\gamma - E_{\gamma_2}\right) \left(E_\gamma - E_{\gamma_3}\right) \left(E_\gamma - E_{\gamma_4}\right)}. \end{aligned} \tag{B299}$$

$$A_6(nnnnn, cccc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} \left(g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma}\right)^2 \delta_{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)^2 \left(E_\gamma - E_{\gamma_2}\right)^2}. \tag{B300}$$

$$A_6(nnnnn, ccnc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_3} g_1^{\gamma_3\gamma} \eta_{\gamma_2\gamma_3}\delta_{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)^2 \left(E_\gamma - E_{\gamma_2}\right) \left(E_\gamma - E_{\gamma_3}\right)}. \tag{B301}$$

$$A_6(nnnnn, cncc; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma_3} g_1^{\gamma_3\gamma_2} g_1^{\gamma_2\gamma} \eta_{\gamma_1\gamma_3}\delta_{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right) \left(E_\gamma - E_{\gamma_2}\right)^2 \left(E_\gamma - E_{\gamma_3}\right)}. \tag{B302}$$

$$A_6(nnnnn, cnnc, kck; t^2\mathrm{e}, t^3\mathrm{e}) \;=\; \sum_{\gamma_1,\gamma_2,\gamma_3} \frac{(-\mathrm{i}t)^2}{2} \frac{\mathrm{e}^{-\mathrm{i}E_\gamma t} g_1^{\gamma\gamma_1} g_1^{\gamma_1\gamma_2} g_1^{\gamma_2\gamma} g_1^{\gamma\gamma_3} g_1^{\gamma_3\gamma_1} g_1^{\gamma_1\gamma} \eta_{\gamma_2\gamma_3}\delta_{\gamma\gamma'}}{\left(E_\gamma - E_{\gamma_1}\right)^2 \left(E_\gamma - E_{\gamma_2}\right) \left(E_\gamma - E_{\gamma_3}\right)}. \tag{B303}$$

$$
\begin{aligned}
&A_6(nnnnn,cnnc,knk;t^2\mathrm{e},t^3\mathrm{e})\\
&=\sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4}\frac{(-\mathrm{i}t)^2}{2}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma_2}g_1^{\gamma_2\gamma}g_1^{\gamma\gamma_3}g_1^{\gamma_3\gamma_4}g_1^{\gamma_4\gamma}\eta_{\gamma_1\gamma_3}\eta_{\gamma_1\gamma_4}\eta_{\gamma_2\gamma_3}\eta_{\gamma_2\gamma_4}\delta_{\gamma\gamma'}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma_2})(E_\gamma-E_{\gamma_3})(E_\gamma-E_{\gamma_4})}.
\end{aligned}
\tag{B304}
$$

Based on above non-zero 91 terms and vanishing 112 terms with $t^2\mathrm{e},t^3\mathrm{e}$ factor parts in all of 203 contraction- and anti contraction- expressions, we can, via the reorganization and summation, obtain the following concise forms:

$$
\begin{aligned}
A_6(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{D}) &= \frac{(-\mathrm{i}t)^2}{2!}\left(G_\gamma^{(3)}\right)^2\mathrm{e}^{-\mathrm{i}E_\gamma t}\delta_{\gamma\gamma'}+\frac{(-\mathrm{i}t)^2}{2!}2G_\gamma^{(2)}G_\gamma^{(4)}\mathrm{e}^{-\mathrm{i}E_\gamma t}\delta_{\gamma\gamma'}\\
&\quad-\frac{(-\mathrm{i}t)^2}{2!}\sum_{\gamma_1}\frac{\left(G_\gamma^{(2)}\right)^2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})^2}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma}\delta_{\gamma\gamma'}.
\end{aligned}
\tag{B305}
$$

$$
A_6(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t},\mathrm{D}) = \frac{(-\mathrm{i}t)^2}{2!}\sum_{\gamma_1}\frac{\left(G_\gamma^{(2)}\right)^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma-E_{\gamma_1})^2}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma}\delta_{\gamma\gamma'}.
\tag{B306}
$$

$$
A_6(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'} t},\mathrm{D}) = 0.
\tag{B307}
$$

$$
\begin{aligned}
A_6(t^2\mathrm{e}^{-\mathrm{i}E_\gamma t},\mathrm{N}) &= \frac{(-\mathrm{i}t)^2}{2!}2G_\gamma^{(2)}G_\gamma^{(3)}\frac{\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma'})}g_1^{\gamma\gamma'}\\
&\quad+\frac{(-\mathrm{i}t)^2}{2!}\sum_{\gamma_1}\frac{\left(G_\gamma^{(2)}\right)^2\mathrm{e}^{-\mathrm{i}E_\gamma t}}{(E_\gamma-E_{\gamma_1})(E_\gamma-E_{\gamma'})}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}.
\end{aligned}
\tag{B308}
$$

$$
A_6(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t},\mathrm{N}) = -\frac{(-\mathrm{i}t)^2}{2!}\sum_{\gamma_1}\frac{\left(G_{\gamma_1}^{(2)}\right)^2\mathrm{e}^{-\mathrm{i}E_{\gamma_1} t}}{(E_\gamma-E_{\gamma_1})(E_{\gamma_1}-E_{\gamma'})}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}.
\tag{B309}
$$

$$
\begin{aligned}
A_6(t^2\mathrm{e}^{-\mathrm{i}E_{\gamma'} t},\mathrm{N}) &= -\frac{(-\mathrm{i}t)^2}{2!}2G_{\gamma'}^{(2)}G_{\gamma'}^{(3)}\frac{\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})}g_1^{\gamma\gamma'}\\
&\quad+\frac{(-\mathrm{i}t)^2}{2!}\sum_{\gamma_1}\frac{\left(G_{\gamma'}^{(2)}\right)^2\mathrm{e}^{-\mathrm{i}E_{\gamma'} t}}{(E_\gamma-E_{\gamma'})(E_{\gamma_1}-E_{\gamma'})}g_1^{\gamma\gamma_1}g_1^{\gamma_1\gamma'}\eta_{\gamma\gamma'}.
\end{aligned}
\tag{B310}
$$

Their forms are indeed the same as expected and can be merged reasonably to the lower order terms in order to obtain the improved forms of perturbed solutions.

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